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OPTIMAL TESTS FOR MULTIVARIATE LOCATION BASED ON INTERDIRECTIONS AND PSEUDO-MAHALANOBIS RANKS

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We propose a family of tests, based on Randles' (1989) concept of interdirections and the ranks of pseudo-Mahalanobis distances computed with respect to a multivariate M-estimator of scatter due to Tyler (1987), for the multivariate one-sample problem under elliptical symmetry. These tests, which generalize the univariate signed-rank tests, are affine-invariant. Depending on the score function considered (van der Waerden, Laplace, ...), they allow for locally asymptotically maximin tests at selected densities (multivariate normal, multivariate double-exponential, ...). Local powers and asymptotic relative efficiencies are derived—with respect to Hotelling's test, Randles' (1989) multivariate sign test, Peters and Randles' (1990) Wilcoxon-type test, and with respect to the Oja median tests. We, moreover, extend to the multivariate setting two famous univariate results: the traditional Chernoff–Savage (1958) property, showing that Hotelling's traditional procedure is uniformly dominated, in the Pitman sense, by the van der Waerden version of our tests, and the celebrated Hodges–Lehmann (1956) ".864 result," providing, for any fixed space dimension k, the lower bound for the asymptotic relative efficiency of Wilcoxon-type tests with respect to Hotelling's.

These asymptotic results are confirmed by a Monte Carlo investigation, and application to a real data set.

1. Introduction and main assumptions. Denote by $(\mathbf{X}_1^{(n)}, \mathbf{X}_2^{(n)}, \ldots, \mathbf{X}_n^{(n)})$ a sequence of k-dimensional i.i.d. observations with elliptically symmetric density centered at $\boldsymbol{\theta}$. Our objective is to test the null hypothesis $\mathcal{H}_0^{(n)}$ under which $\boldsymbol{\theta}$ is equal to some given $\boldsymbol{\theta}_0$, the elliptically symmetric density remaining otherwise unspecified.

The classical procedure for this problem is the Hotelling test, which is optimal under Gaussian densities, and rejects the null hypothesis whenever

$$\frac{n-k}{k(n-1)} T^{2(n)} := \frac{n(n-k)}{k(n-1)} (\bar{\mathbf{X}}^{(n)} - \boldsymbol{\theta}_0)' (\mathbf{S}^{(n)})^{-1} (\bar{\mathbf{X}}^{(n)} - \boldsymbol{\theta}_0),$$

where $\bar{\mathbf{X}}^{(n)} := \frac{1}{n}\sum_{i=1}^n \mathbf{X}_i^{(n)}$ and $\mathbf{S}^{(n)} := \frac{1}{n-1}\sum_{i=1}^n (\mathbf{X}_i^{(n)} - \bar{\mathbf{X}}^{(n)})(\mathbf{X}_i^{(n)} - \bar{\mathbf{X}}^{(n)})'$, exceeds the $(1-\alpha)$-quantile $F_{k,n-k;1-\alpha}$ of a Fisher distribution with k and $n-k$ degrees of freedom.

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Many nonparametric methods have been developed, as a reaction to the Gaussian approach of Hotelling's test, with the objective of extending to the multivariate context the classical univariate rank and signed-rank techniques. Essentially, these methods belong to three main groups. The first of these groups relies on componentwise rankings [see, e.g., the monograph by Puri and Sen (1971)], but suffers from a severe lack of invariance with respect to affine transformations, which has been the main motivation for the other two approaches. The second group [Möttönen et al. (1995, 1997, 1998); Hettmansperger et al. (1994, 1997); see Oja (1999) for a recent review] is closely related with the spatial signs and ranks, and with the so-called Oja median, the third one [Randles (1989); Peters and Randles (1990); Jan and Randles (1994)] with the concept of interdirections.

The present paper belongs to this third group. However, in addition to the robustness and invariance issues discussed in the work by Randles and his coauthors, we put more emphasis on optimality arguments a la Le Cam–Hájek. Mainly, we show that locally asymptotically optimal tests can be based on interdirections complemented by the ranks of *pseudo-Mahalanobis distances*; *pseudo* is used here to emphasize the fact that distances are taken with respect to an estimated scatter matrix—not necessarily the empirical covariance matrix, as in classical Mahalanobis distances. The Le Cam theory, in addition, allows for computing local powers and asymptotic relative efficiencies.

Hotelling's statistic, as a quadratic form in the observed mean, can be expressed as a sum of scalar products (i.e., moduli and cosines) between couples of observations. Our test statistic (5) is also a quadratic form, but involving ranks of moduli, and invariant evaluation of cosines based on Randles' interdirections. While sharing Hotelling's strict affine invariance property, our procedure offers a number of advantages:

(i) broader validity (even in the absence of second order moments),
(ii) better robustness features,
(iii) increased power (see the Chernoff–Savage property in Proposition 6).

As in Peters and Randles (1990) and Jan and Randles (1994), we throughout assume that the common density $\underline{f}$ of the observations satisfies the following assumption.

ASSUMPTION (A1). (Elliptical symmetry.) There exist $\boldsymbol{\theta} \in \mathbb{R}^k$, a symmetric positive definite matrix $\boldsymbol{\Sigma}$ and a function $f : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ such that $\int_0^\infty r^{k-1} f(r)\, dr < \infty$ and

$$\underline{f}(\mathbf{x}; \boldsymbol{\theta}, \boldsymbol{\Sigma}, f) := c_{k,f} \frac{1}{(\det \boldsymbol{\Sigma})^{1/2}} f(\|\mathbf{x}\|_{\boldsymbol{\theta},\boldsymbol{\Sigma}}), \qquad \mathbf{x} \in \mathbb{R}^k, \tag{1}$$

where $\|\mathbf{x}\|_{\boldsymbol{\theta},\boldsymbol{\Sigma}} := ((\mathbf{x} - \boldsymbol{\theta})' \boldsymbol{\Sigma}^{-1} (\mathbf{x} - \boldsymbol{\theta}))^{1/2}$ denotes the distance between $\mathbf{x}$ and $\boldsymbol{\theta}$ in the metric associated with $\boldsymbol{\Sigma}$. The constant $c_{k,f}$ is the normalization factor

$(\omega_k \, \mu_{k-1;f})^{-1}$, where ω_k stands for the $(k-1)$-dimensional Lebesgue measure of the unit sphere $\mathcal{S}^{k-1} \subset \mathbb{R}^k$ and $\mu_{\ell;f} := \int_0^\infty r^\ell f(r)\,dr$.

Note that $\mathbf{\Sigma}$ need not be the covariance matrix of the observations; moreover, $\mathbf{\Sigma}$ and f are only identified up to an arbitrary scale factor. This will not be a problem since we just need the multivariate scatter matrix $c\mathbf{\Sigma}$ (for some arbitrary $c > 0$), not $\mathbf{\Sigma}$ itself [see Assumption (A4) below].

Whenever we consider Hotelling's test, or more generally, whenever we need finite second-order moments, Assumption (A1) has to be strengthened into the following one:

ASSUMPTION (A1′). Same as Assumption (A1), but we further assume that $\mu_{k+1;f} < \infty$.

The function f appearing in (1) will be called "radial density," though it does not necessarily integrate to one. This terminology is justified by the fact that if a random vector $\mathbf{X}$ has density $\underline{f}(\cdot;\boldsymbol{\theta},\mathbf{\Sigma},f)$, then $\|\mathbf{\Sigma}^{-1/2}(\mathbf{X}-\boldsymbol{\theta})\|$, where $\mathbf{\Sigma}^{1/2}$ denotes an arbitrary square root of $\mathbf{\Sigma}$, has density $\tilde{f}_k(r) := (\mu_{k-1;f})^{-1} r^{k-1} f(r) I_{[r>0]}$. It should be stressed that $\mathbf{\Sigma}^{1/2}$ is defined up to an orthogonal transformation, so that we always may assume, without loss of generality, that $\mathbf{\Sigma}^{-1/2}(\mathbf{X}_1^{(n)}-\boldsymbol{\theta})/\|\mathbf{\Sigma}^{-1/2}(\mathbf{X}_1^{(n)}-\boldsymbol{\theta})\|$ is of the form $(1,0,\ldots,0)'$. In the sequel, $\tilde{F}_k$ stands for the distribution function associated with $\tilde{f}_k$.

The hypothesis under which the observations have joint density $\prod_{i=1}^n \underline{f}(\mathbf{X}_i^{(n)};\boldsymbol{\theta},\mathbf{\Sigma},f)$ will be denoted as $\mathcal{H}^{(n)}(\boldsymbol{\theta},\mathbf{\Sigma},f)$. We also denote by $\mathcal{H}^{(n)}(\boldsymbol{\theta},\mathbf{\Sigma},\cdot)$, $\mathcal{H}^{(n)}(\boldsymbol{\theta},\cdot,f)$ and $\mathcal{H}^{(n)}(\boldsymbol{\theta},\cdot,\cdot)$ the hypotheses $\bigcup_f \mathcal{H}^{(n)}(\boldsymbol{\theta},\mathbf{\Sigma},f)$, $\bigcup_{\mathbf{\Sigma}} \mathcal{H}^{(n)}(\boldsymbol{\theta},\mathbf{\Sigma},f)$ and $\bigcup_{\mathbf{\Sigma}}\bigcup_f \mathcal{H}^{(n)}(\boldsymbol{\theta},\mathbf{\Sigma},f)$, respectively, where unions are taken over the largest sets that are compatible with the assumptions. Under this notation, the testing problem we are interested in is

$$\mathcal{H}_0^{(n)} = \mathcal{H}^{(n)}(\boldsymbol{\theta}_0,\cdot,\cdot),\qquad \mathcal{H}_1^{(n)} = \bigcup_{\boldsymbol{\theta}\neq\boldsymbol{\theta}_0} \mathcal{H}^{(n)}(\boldsymbol{\theta},\cdot,\cdot). \tag{2}$$

Our approach is based on the local asymptotic normality (LAN) structure, for fixed $\mathbf{\Sigma}$ and f, of the family of densities of the form (1). LAN of course requires some further regularity assumptions on the radial density f, a neat statement of which calls for some clarification of the concept of quadratic mean differentiability in a multivariate context.

Consider the measure space $(\Omega,\mathbb{B}_\Omega^k,\lambda)$, where λ is some measure on the open subset $\Omega \subset \mathbb{R}^k$ equipped with its Borel σ-field $\mathbb{B}_\Omega^k$. Denote by $L^2(\Omega,\lambda)$ the space of measurable functions $h:\Omega\to\mathbb{R}$ satisfying $\int_\Omega [h(\mathbf{x})]^2\,d\lambda(\mathbf{x}) < \infty$.

In particular, consider the space $L^2(\mathbb{R}_0^+, \mu_\ell)$ of square-integrable functions w.r.t. the Lebesgue measure with weight r^ℓ on $\mathbb{R}_0^+$, that is, the space of measurable functions $h : \mathbb{R}_0^+ \to \mathbb{R}$ satisfying $\int_0^\infty [h(r)]^2 r^\ell \, dr < \infty$. Recall that $g \in L^2(\Omega, \lambda)$ admits a *weak partial derivative* T_i w.r.t. to the ith variable iff

$$\int_\Omega g(\mathbf{x}) \, \partial_i \varphi(\mathbf{x}) \, d\mathbf{x} = -\int_\Omega T_i(\mathbf{x}) \, \varphi(\mathbf{x}) \, d\mathbf{x},$$

for all functions $\varphi \in C_0^\infty(\Omega)$, that is, for all infinitely differentiable (in the classical sense) compactly supported function over Ω. If T_i exists for all i, the gradient $\mathbf{T} := (T_1, \dots, T_k)$ is also called the *derivative of g in the sense of distributions* in $L^2(\Omega, \lambda)$. If, moreover, $\mathbf{T}$ itself is in $L^2(\Omega, \lambda)$, then g belongs to $W^{1,2}(\Omega, \lambda)$, the Sobolev space of order 1 on $L^2(\Omega, \lambda)$. Equipped with the norm $\|g\|_{W^{1,2}(\Omega,\lambda)} := (\|g\|^2_{L^2(\Omega,\lambda)} + \sum_{i=1}^k \|T_i\|^2_{L^2(\Omega,\lambda)})^{1/2}$, $W^{1,2}(\Omega, \lambda)$ is a Banach space [see Adams (1975), Chapter 3, page 44 for details]. For the sake of simplicity, we will write $L^2(\Omega)$ and $W^{1,2}(\Omega)$, when λ is the Lebesgue measure on Ω.

The minimal assumption we are making on the radial density f is:

ASSUMPTION (A2). The square root $f^{1/2}$ of the radial density f is in $W^{1,2}(\mathbb{R}_0^+, \mu_{k-1})$; denote by $(f^{1/2})'$ its weak derivative in $L^2(\mathbb{R}_0^+, \mu_{k-1})$.

The following result, which elucidates some aspects of quadratic mean differentiablity in this multivariate context, shows that Assumption (A2) is strictly equivalent to the quadratic mean differentiability of $\mathbf{x} \mapsto f^{1/2}(\|\mathbf{x}\|)$. Its advantage over the latter, (involving $f^{1/2}$ as a function over $\mathbb{R}^k$) is that it treats $f^{1/2}$ as a function defined over $\mathbb{R}_0^+$.

PROPOSITION 1. *The function $g : \mathbf{x} \mapsto h(\|\mathbf{x}\|)$ is differentiable in quadratic mean over $\mathbb{R}^k$ if and only if $h \in W^{1,2}(\mathbb{R}_0^+, \mu_{k-1})$. In that case, the quadratic mean gradient $\mathbf{D}g(\mathbf{x})$ can be taken as $h'(\|\mathbf{x}\|)\mathbf{x}/\|\mathbf{x}\|$ in $L^2(\mathbb{R}^k)$, where h' stands for the weak derivative of h in $L^2(\mathbb{R}_0^+, \mu_{k-1})$.*

The proof of this proposition, which is of independent interest, is given in Appendix A.

Letting $\varphi_f := -2\frac{(f^{1/2})'}{f^{1/2}}$, Assumption (A2) ensures the finiteness of the *radial Fisher information* $\mathcal{I}_{k,f} := \int_0^1 \varphi_f^2(\tilde{F}_k^{-1}(u))\, du$. Under this form, (A2) will be sufficient for LAN. Whenever ranks come into the picture and φ_f is to be used as a score-generating function, we will consider the following, slightly stronger version of Assumption (A2):

ASSUMPTION (A3). Same as (A2) but, in addition, $\varphi_f(u)$ satisfies $\int_0^1 |\varphi_f(\tilde{F}_k^{-1}(u))|^{2+\delta}\, du < \infty$ for some $\delta > 0$, is continuous, and can be expressed as the difference of two monotone increasing functions.

Assumption (A2) unfortunately is not easy to check; the following sufficient condition brings it back to more familiar univariate concepts, while covering most cases of practical interest.

ASSUMPTION (A2$'$). The radial density f is absolutely continuous, with a.e. derivative f', and $(f^{1/2})' := \frac{f'}{2f^{1/2}}$ is in $L^2(\mathbb{R}_0^+, \mu_{k-1})$.

Finally, the matrix $\mathbf{\Sigma}$ in practice is not known, and has to be estimated from the observations. We assume that a sequence of statistics $\hat{\mathbf{\Sigma}}^{(n)}$ is available, with the following properties.

ASSUMPTION (A4). The sequence $\hat{\mathbf{\Sigma}}^{(n)}$ is invariant under permutations and reflections with respect to the origin in $\mathbb{R}^k$ of the observations; $\sqrt{n}(\hat{\mathbf{\Sigma}}^{(n)} - c\,\mathbf{\Sigma})$ is $O_{\mathrm{P}}(1)$ under $\mathcal{H}^{(n)}(\mathbf{0}, \mathbf{\Sigma}, f)$ for some $c > 0$. The corresponding *pseudo-Mahalanobis distances* $(\mathbf{X}_i^{(n)\prime}(\hat{\mathbf{\Sigma}}^{(n)})^{-1}\mathbf{X}_i^{(n)})^{1/2}$ are *quasi-affine-invariant* in the sense that, if $\mathbf{Y}_i^{(n)} = \mathbf{M}\mathbf{X}_i^{(n)}$ for all i, where $\mathbf{M}$ is an arbitrary nonsingular $k \times k$ matrix,

$$\big(\mathbf{Y}_i^{(n)\prime}(\hat{\mathbf{\Sigma}}_{\mathbf{y}}^{(n)})^{-1}\mathbf{Y}_i^{(n)}\big)^{1/2} = d \times \big(\mathbf{X}_i^{(n)\prime}(\hat{\mathbf{\Sigma}}_{\mathbf{x}}^{(n)})^{-1}\mathbf{X}_i^{(n)}\big)^{1/2},$$

for some positive scalar d that may depend on $\mathbf{M}$ and the sample $(\mathbf{X}_1^{(n)}, \dots, \mathbf{X}_n^{(n)})$, but not on index i.

Note that quasi-affine-invariance of pseudo-Mahalanobis distances implies strict affine-invariance of their ranks.

If $\hat{\mathbf{\Sigma}}^{(n)}$ is computed from the residuals $\mathbf{X}_1^{(n)} - \boldsymbol{\theta}_0, \dots, \mathbf{X}_n^{(n)} - \boldsymbol{\theta}_0$, Assumption (A4) also yields a sequence of estimators for $c\mathbf{\Sigma}$ that is $\sqrt{n}$-consistent under $\mathcal{H}^{(n)}(\boldsymbol{\theta}_0, \mathbf{\Sigma}, f)$, invariant under the permutations and reflections (with respect to $\boldsymbol{\theta}_0$) of the observations, and defining pseudo-Mahalanobis distances between $\mathbf{X}_i^{(n)}$ and $\boldsymbol{\theta}_0$ that are quasi-invariant under linear transformations acting on the residuals. For the sake of simplicity, let $\boldsymbol{\theta}_0 = \mathbf{0}$ in the rest of this section.

Under Assumption (A1$'$), $\underline{f}$ has finite second moments, and we may use for $\hat{\mathbf{\Sigma}}^{(n)}$ the empirical covariance matrix $n^{-1}\sum_{i=1}^n \mathbf{X}_i^{(n)}(\mathbf{X}_i^{(n)})'$, which is consistent for $\mathrm{Cov}[\mathbf{X}] = \frac{\mu_{k+1;f}}{k\mu_{k-1;f}}\mathbf{\Sigma}$. Pseudo-Mahalanobis distances then coincide with the classical ones, which are of course strictly affine-invariant.

If [as in Assumption (A1)] no assumption is made about the moments of the radial density, the empirical covariance matrix may not be root-n consistent. Other affine-equivariant estimators of scatter then are to be considered, such as the following one, which was proposed by Tyler (1987). For the k-dimensional sample $(\mathbf{X}_1^{(n)}, \mathbf{X}_2^{(n)}, \dots, \mathbf{X}_n^{(n)})$, define the $k \times k$ matrix $\mathbf{A}^{(n)}$ as the [unique for $n > k(k-1)$;

see Tyler (1987)] upper triangular matrix with positive diagonal elements and a "1" in the upper left corner that satisfies

$$\frac{1}{n}\sum_{i=1}^{n}\left(\frac{\mathbf{A}^{(n)}\mathbf{X}_i^{(n)}}{\|\mathbf{A}^{(n)}\mathbf{X}_i^{(n)}\|}\right)\left(\frac{\mathbf{A}^{(n)}\mathbf{X}_i^{(n)}}{\|\mathbf{A}^{(n)}\mathbf{X}_i^{(n)}\|}\right)' = \frac{1}{k}\mathbf{I}_k. \tag{3}$$

This matrix $\mathbf{A}^{(n)}$ is such that the covariance structure of the n-tuple

$$\left(\frac{\mathbf{A}^{(n)}\mathbf{X}_1^{(n)}}{\|\mathbf{A}^{(n)}\mathbf{X}_1^{(n)}\|}, \dots, \frac{\mathbf{A}^{(n)}\mathbf{X}_n^{(n)}}{\|\mathbf{A}^{(n)}\mathbf{X}_n^{(n)}\|}\right)$$

is that of a sample that is uniformly distributed on the unit sphere $\mathcal{S}^{k-1}$. Randles showed that, for any nonsingular $k \times k$ matrix $\mathbf{M}$, letting $\mathbf{Y}_i^{(n)} = \mathbf{M}\mathbf{X}_i^{(n)}$ for all i,

$$\mathbf{M}'(\mathbf{A}_{\mathbf{y}}^{(n)})'\mathbf{A}_{\mathbf{y}}^{(n)}\mathbf{M} = d(\mathbf{A}_{\mathbf{x}}^{(n)})'\mathbf{A}_{\mathbf{x}}^{(n)}$$

for some positive scalar d that may depend on $\mathbf{M}$ and on the sample $(\mathbf{X}_1^{(n)}, \dots, \mathbf{X}_n^{(n)})$ [see Randles (2000)]. This shows that, if we take $\hat{\mathbf{\Sigma}}_{\mathrm{Tyl}}^{(n)} := [(\mathbf{A}^{(n)})'\mathbf{A}^{(n)}]^{-1}$, the resulting pseudo-Mahalanobis distances are quasi-affine-invariant in the sense of Assumption (A4). See Tyler (1987) or Randles (2000) for the consistency properties of $\hat{\mathbf{\Sigma}}_{\mathrm{Tyl}}^{(n)}$, and for a performant iterative computation scheme.

2. Local asymptotic normality, parametric optimality and group invariance.

2.1. *Local asymptotic normality* (*LAN*). Local asymptotic normality, for given $\mathbf{\Sigma}$ and f, takes the following form.

PROPOSITION 2. *Assume that Assumptions* (A1) *and* (A2) *hold. Let* $d_i^{(n)}(\boldsymbol{\theta}, \mathbf{\Sigma}) := \|\mathbf{X}_i^{(n)}\|_{\boldsymbol{\theta},\mathbf{\Sigma}}$ *and* $\mathbf{U}_i^{(n)}(\boldsymbol{\theta}, \mathbf{\Sigma}) := \mathbf{\Sigma}^{-1/2}(\mathbf{X}_i^{(n)} - \boldsymbol{\theta})/d_i^{(n)}(\boldsymbol{\theta}, \mathbf{\Sigma})$ *for all* $i = 1, \dots, n$. *Denote by* $(\boldsymbol{\tau}^{(n)})$, $\boldsymbol{\tau}^{(n)} \in \mathbb{R}^k$, *an arbitrary sequence such that* $\sup_n(\boldsymbol{\tau}^{(n)})'\boldsymbol{\tau}^{(n)} < \infty$. *Then, the logarithm of the likelihood ratio associated with the sequence of local alternatives* $\mathcal{H}^{(n)}(\boldsymbol{\theta} + n^{-1/2}\boldsymbol{\tau}^{(n)}, \mathbf{\Sigma}, f)$ *with respect to* $\mathcal{H}^{(n)}(\boldsymbol{\theta}, \mathbf{\Sigma}, f)$ *is such that*

$$L_{\boldsymbol{\theta}+n^{-1/2}\boldsymbol{\tau}^{(n)}/\boldsymbol{\theta};\mathbf{\Sigma},f}^{(n)}(\mathbf{X}^{(n)}) = (\boldsymbol{\tau}^{(n)})'\mathbf{\Delta}_{\mathbf{\Sigma},f}^{(n)}(\boldsymbol{\theta}) - \tfrac{1}{2}(\boldsymbol{\tau}^{(n)})'\mathbf{\Gamma}_{\mathbf{\Sigma},f}\boldsymbol{\tau}^{(n)} + o_{\mathrm{P}}(1),$$

as $n \to \infty$, *under* $\mathcal{H}^{(n)}(\boldsymbol{\theta}, \mathbf{\Sigma}, f)$, *with the central sequence*

$$\mathbf{\Delta}_{\mathbf{\Sigma},f}^{(n)}(\boldsymbol{\theta}) := n^{-1/2}\sum_{i=1}^{n}\varphi_f\big(d_i^{(n)}(\boldsymbol{\theta}, \mathbf{\Sigma})\big)\mathbf{\Sigma}^{-1/2}\mathbf{U}_i^{(n)}(\boldsymbol{\theta}, \mathbf{\Sigma}),$$

and $\mathbf{\Gamma}_{\mathbf{\Sigma},f} := \frac{1}{k}\mathcal{I}_{k,f}\mathbf{\Sigma}^{-1}$. *Moreover,* $\mathbf{\Delta}_{\mathbf{\Sigma},f}^{(n)}(\boldsymbol{\theta})$, *still under* $\mathcal{H}^{(n)}(\boldsymbol{\theta}, \mathbf{\Sigma}, f)$, *is asymptotically* $\mathcal{N}_k(\mathbf{0}, \mathbf{\Gamma}_{\mathbf{\Sigma},f})$.

PROOF. The proof consists in checking that the assumptions of Swensen [(1985), Lemma 1] are satisfied under Assumptions (A1) and (A2). Most of them however follow quite routinely from classical arguments once it is shown that $\mathbf{x} \mapsto f^{1/2}(\|\mathbf{x}\|)$ is differentiable in quadratic mean. This is an immediate consequence of Proposition 1, which we prove in Appendix A. □

2.2. *Parametric optimality.* The form of locally and asymptotically optimal testing procedures for testing $\boldsymbol{\theta} = \boldsymbol{\theta}_0$ under specified $\boldsymbol{\Sigma}$ and f readily follows from LAN [see Le Cam (1986), Section 11.9]. The test rejecting $\mathcal{H}^{(n)}(\boldsymbol{\theta}_0, \boldsymbol{\Sigma}, f)$ whenever the quadratic statistic

$$\begin{aligned} Q^{(n)}_{\boldsymbol{\Sigma},f}(\boldsymbol{\theta}_0) &:= \boldsymbol{\Delta}^{(n)\prime}_{\boldsymbol{\Sigma},f}(\boldsymbol{\theta}_0)\boldsymbol{\Gamma}^{-1}_{\boldsymbol{\Sigma},f}\boldsymbol{\Delta}^{(n)}_{\boldsymbol{\Sigma},f}(\boldsymbol{\theta}_0) \\ &= \frac{k}{n\,\mathcal{I}_{k,f}} \sum_{i,j=1}^{n} \varphi_f\big(d_i^{(n)}(\boldsymbol{\theta}_0, \boldsymbol{\Sigma})\big)\varphi_f\big(d_j^{(n)}(\boldsymbol{\theta}_0, \boldsymbol{\Sigma})\big) \\ &\qquad \times \big(\mathbf{U}_i^{(n)}(\boldsymbol{\theta}_0, \boldsymbol{\Sigma})\big)'\mathbf{U}_j^{(n)}(\boldsymbol{\theta}_0, \boldsymbol{\Sigma}) \end{aligned}$$

exceeds the $(1-\alpha)$-quantile $\chi^2_{k,1-\alpha}$ of a chi-square distribution with k degrees of freedom is locally asymptotically maximin at asymptotic level α, for $\mathcal{H}^{(n)}(\boldsymbol{\theta}_0, \boldsymbol{\Sigma}, f)$ under fixed $\boldsymbol{\Sigma}$ and f.

2.3. *Group invariance, interdirections and pseudo-Mahalanobis ranks.* In this section, we briefly review the invariance properties of the problem under study, and introduce the invariant statistics—interdirections and (pseudo-)Mahalanobis ranks—to be used in the sequel. Denote by $\mathbf{Z}_i^{(n)}(\boldsymbol{\theta}, \boldsymbol{\Sigma})$ $(i = 1, \dots, n)$ the residuals $\boldsymbol{\Sigma}^{-1/2}(\mathbf{X}_i^{(n)} - \boldsymbol{\theta})$. Note that, under $\mathcal{H}^{(n)}(\boldsymbol{\theta}, \boldsymbol{\Sigma}, \cdot)$, the vectors $\mathbf{U}_i^{(n)}(\boldsymbol{\theta}, \boldsymbol{\Sigma}) = \mathbf{Z}_i^{(n)}(\boldsymbol{\theta}, \boldsymbol{\Sigma})/\|\mathbf{Z}_i^{(n)}(\boldsymbol{\theta}, \boldsymbol{\Sigma})\|$ are independent and uniformly distributed over the unit sphere $\mathcal{S}^{k-1}$. The notation $\hat{\mathbf{Z}}_i^{(n)}(\boldsymbol{\theta})$ will be used for the residuals $\mathbf{Z}_i^{(n)}(\boldsymbol{\theta}, \hat{\boldsymbol{\Sigma}})$ associated with the estimator $\hat{\boldsymbol{\Sigma}}$ considered in (A4).

The interdirection $c_{ij}^{(n)}(\boldsymbol{\theta})$ associated with the pair of residuals $(\hat{\mathbf{Z}}_i^{(n)}(\boldsymbol{\theta}), \hat{\mathbf{Z}}_j^{(n)}(\boldsymbol{\theta}))$ in the n-tuple of residuals $\hat{\mathbf{Z}}_1^{(n)}(\boldsymbol{\theta}), \dots, \hat{\mathbf{Z}}_n^{(n)}(\boldsymbol{\theta})$ has been defined [Randles (1989)] as the number of hyperplanes in $\mathbb{R}^k$ passing through the origin and $k-1$ out of the $n-2$ points $\hat{\mathbf{Z}}_1^{(n)}(\boldsymbol{\theta}), \dots, \hat{\mathbf{Z}}_{i-1}^{(n)}(\boldsymbol{\theta}), \hat{\mathbf{Z}}_{i+1}^{(n)}(\boldsymbol{\theta}), \dots, \hat{\mathbf{Z}}_{j-1}^{(n)}(\boldsymbol{\theta}), \hat{\mathbf{Z}}_{j+1}^{(n)}(\boldsymbol{\theta}), \dots, \hat{\mathbf{Z}}_n^{(n)}(\boldsymbol{\theta})$, that are *separating* $\hat{\mathbf{Z}}_i^{(n)}(\boldsymbol{\theta})$ and $\hat{\mathbf{Z}}_j^{(n)}(\boldsymbol{\theta})$: obviously, $0 \le c_{ij}^{(n)}(\boldsymbol{\theta}) \le \binom{n-2}{k-1}$. Interdirections are invariant under linear transformations, so that they can be computed from the residuals $\hat{\mathbf{Z}}_i^{(n)}$ just as easily as from the residuals $\mathbf{Z}_i^{(n)}$, or from the centered observations $\mathbf{X}_i^{(n)} - \boldsymbol{\theta}$ themselves.

For the same reasons as in Randles (1989), we rather consider the *normalized* interdirections

$$p_{ij}^{(n)}(\boldsymbol{\theta}) := \begin{cases} \dfrac{c_{ij}^{(n)}(\boldsymbol{\theta}) + d_k^{(n)}}{\binom{n}{k-1}}, & \text{if } i \neq j, \\ 0, & \text{if } i = j, \end{cases}$$

where $d_k^{(n)} := \frac{1}{2}[\binom{n}{k-1} - \binom{n-2}{k-1}]$. Interdirections provide affine-invariant estimations of the Euclidean angles between residuals $\mathbf{Z}_j^{(n)}(\boldsymbol{\theta}, \boldsymbol{\Sigma})$, that is, they estimate the quantities

$$\pi^{-1} \arccos\bigl[(\mathbf{U}_i^{(n)}(\boldsymbol{\theta}, \boldsymbol{\Sigma}))' \mathbf{U}_j^{(n)}(\boldsymbol{\theta}, \boldsymbol{\Sigma})\bigr].$$

The following consistency result is stated in Randles (1989), Peters and Randles (1990) and Jan and Randles (1994). We restate it here, with a proof based on a U-statistic representation.

LEMMA 1. *Let* $(\mathbf{X}_1, \mathbf{X}_2, \ldots)$ *be an i.i.d. process of* k*-variate random vectors with spherically symmetric density. For any fixed* $\mathbf{v}$ *and* $\mathbf{w}$ *in* $\mathbb{R}^k$, *denote by* $\alpha(\mathbf{v}, \mathbf{w}) := \arccos(\mathbf{v}'\mathbf{w}/(\|\mathbf{v}\|\|\mathbf{w}\|))$ *the angle between* $\mathbf{v}$ *and* $\mathbf{w}$, *and by* $c^{(n)}(\mathbf{v}, \mathbf{w})$ *the interdirection associated with* $\mathbf{v}$ *and* $\mathbf{w}$ *in the sample* $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n$. *Then,* $c^{(n)}(\mathbf{v}, \mathbf{w})/\binom{n}{k-1}$ *converges in quadratic mean to* $\pi^{-1}\alpha(\mathbf{v}, \mathbf{w})$ *as* $n \to \infty$.

PROOF. The lemma is trivial for $k = 1$, and readily follows from the law of large numbers for $k = 2$. For $k > 2$, define $\mathcal{Q} := \{\mathbf{q} = (i_1, i_2, \ldots, i_{k-1}) \mid 1 \leq i_1 < i_2 < \cdots < i_{k-1} \leq n\}$ and, as in Oja (1999), let $\mathbf{e_q}$ be the k-vector whose components are the cofactors of $\mathbf{x}$ in the $(k \times k)$-array $(\mathbf{X}_{i_1} \mathbf{X}_{i_2} \cdots \mathbf{X}_{i_{k-1}} \mathbf{x})$. Note that $\mathbf{e_q}$ is orthogonal to the hyperplane that goes through $\mathbf{X}_{i_1}, \mathbf{X}_{i_2}, \ldots, \mathbf{X}_{i_{k-1}}$ and the origin in $\mathbb{R}^k$. With this notation,

$$\binom{n}{k-1}^{-1} c^{(n)}(\mathbf{v}, \mathbf{w}) = \binom{n}{k-1}^{-1} \sum_{\mathbf{q} \in \mathcal{Q}} g(\mathbf{X}_{i_1}, \ldots, \mathbf{X}_{i_{k-1}}),$$

where $g(\mathbf{X}_{i_1}, \ldots, \mathbf{X}_{i_{k-1}}) := (1 - \operatorname{sgn}(\mathbf{e_q}'\mathbf{v})\operatorname{sgn}(\mathbf{e_q}'\mathbf{w}))/2$ is symmetric in its arguments [$\operatorname{sgn}(z) := I[z > 0] - I[z < 0]$ stands for the sign function]. Thus, $c^{(n)}(\mathbf{v}, \mathbf{w})/\binom{n}{k-1}$ is the U-statistic with kernel g and expectation $\mathrm{E}[g(\mathbf{X}_{i_1}, \ldots, \mathbf{X}_{i_{k-1}})]$. Since $\mathrm{E}[g^2(\mathbf{X}_{i_1}, \ldots, \mathbf{X}_{i_{k-1}})]$ is finite, this implies that $c^{(n)}(\mathbf{v}, \mathbf{w})/\binom{n}{k-1}$ converges in quadratic mean to $\mathrm{E}[g(\mathbf{X}_{i_1}, \ldots, \mathbf{X}_{i_{k-1}})]$ as $n \to +\infty$.

In order to complete the proof, we now show that $\mathrm{E}[g(\mathbf{X}_{i_1}, \ldots, \mathbf{X}_{i_{k-1}})] = \pi^{-1}\alpha(\mathbf{v}, \mathbf{w})$. Consider the canonical basis $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_k\}$ in $\mathbb{R}^k$. Without loss of generality, we may suppose that $\mathbf{v} = \mathbf{e}_1$ and that $\mathbf{w} = \lambda\mathbf{e}_1 + \mu\mathbf{e}_2$ for some $\lambda, \mu \in \mathbb{R}$. Note that $\mathrm{E}[g(\mathbf{X}_{i_1}, \ldots, \mathbf{X}_{i_{k-1}})] = \mathrm{P}[\operatorname{sgn}(\mathbf{e_q}'\mathbf{v}) = -\operatorname{sgn}(\mathbf{e_q}'\mathbf{w})] = \mathrm{P}[((\mathbf{e_q})_1, (\mathbf{e_q})_2) \in A]$, where A is the hatched region of $\mathbb{R}^2 \subset \mathbb{R}^k$ in Figure 1.

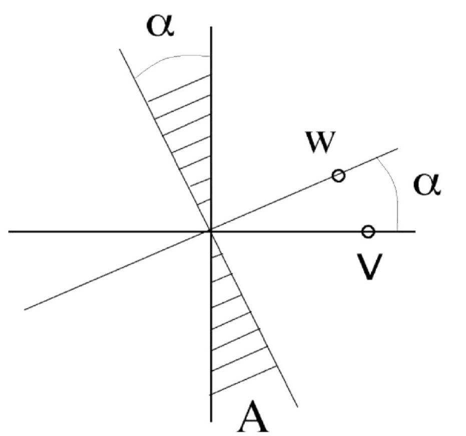

FIG. 1. *Proof of Lemma* 1.

Consider some invertible $k \times k$ matrix $\mathbf{M}$ and denote by $\mathbf{e_q}^\star(\mathbf{M})$ the vector $\mathbf{e_q}$ computed from the transformed observations $\mathbf{MX}_{i_1}, \ldots, \mathbf{MX}_{i_{k-1}}$. Then, for all k-dimensional vectors $\mathbf{x}$, we have $\mathbf{e_q}^\star(\mathbf{M})'\mathbf{x} = \det(\mathbf{MX}_{i_1}, \ldots, \mathbf{MX}_{i_{k-1}}, \mathbf{x}) = (\det \mathbf{M})\mathbf{e_q}'\mathbf{M}^{-1}\mathbf{x}$, so that $\mathbf{e_q}^\star(\mathbf{M}) = (\det \mathbf{M})(\mathbf{M}^{-1})'\mathbf{e_q}$ (incidently, note that this proves that interdirections are affine-invariant). This entails that $((\mathbf{e_q})_1, (\mathbf{e_q})_2)$ is spherically symmetric in $\mathbb{R}^2$. Indeed, letting $\mathbf{O}$ denote an arbitrary orthogonal 2×2 matrix with $\det \mathbf{O} = 1$, define

$$\tilde{\mathbf{O}} := \begin{pmatrix} \mathbf{O} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{k-2} \end{pmatrix},$$

where $\mathbf{I}_{k-2}$ denotes the $(k-2)$-dimensional identity matrix. Then, from the spherical symmetry of the $\mathbf{X}_i$'s, it is clear that $\mathbf{e_q} =_{\mathrm{d}} \mathbf{e_q}^\star(\tilde{\mathbf{O}}) = \tilde{\mathbf{O}}\mathbf{e_q}$, so that $((\mathbf{e_q})_1, (\mathbf{e_q})_2) =_{\mathrm{d}} \mathbf{O}((\mathbf{e_q})_1, (\mathbf{e_q})_2)$ (where $=_{\mathrm{d}}$ stands for equality in distribution). This proves that $((\mathbf{e_q})_1, (\mathbf{e_q})_2)$ is indeed spherically symmetric, which implies that $\mathrm{P}[((\mathbf{e_q})_1, (\mathbf{e_q})_2) \in A] = \pi^{-1}\alpha(\mathbf{v}, \mathbf{w})$. □

The ranks of pseudo-Mahalanobis distances between the observations and their center $\boldsymbol{\theta}$ are the other main tool used in this paper. Let $R_i^{(n)}(\boldsymbol{\theta}, \boldsymbol{\Sigma})$ denote the rank of $d_i^{(n)}(\boldsymbol{\theta}, \boldsymbol{\Sigma})$ among the distances $d_1^{(n)}(\boldsymbol{\theta}, \boldsymbol{\Sigma}), \ldots, d_n^{(n)}(\boldsymbol{\theta}, \boldsymbol{\Sigma})$; write $\hat{R}_i^{(n)}(\boldsymbol{\theta})$ and $\hat{d}_i^{(n)}(\boldsymbol{\theta})$ for $R_i^{(n)}(\boldsymbol{\theta}, \hat{\boldsymbol{\Sigma}})$ and $d_i^{(n)}(\boldsymbol{\theta}, \hat{\boldsymbol{\Sigma}})$, respectively, where $\hat{\boldsymbol{\Sigma}}$ is the estimator considered in (A4). It will be convenient to refer to $\hat{R}_i^{(n)}(\boldsymbol{\theta})$ as the *pseudo-Mahalanobis rank* of $\mathbf{X}_i^{(n)} - \boldsymbol{\theta}$. The following result is proven in Peters and Randles (1990).

LEMMA 2 [Peters and Randles (1990)]. *For all* $i \in \mathbb{N}$, $(\hat{R}_i^{(n)}(\boldsymbol{\theta}) - R_i^{(n)}(\boldsymbol{\theta}, \boldsymbol{\Sigma}))/(n+1)$ *is* $o_{\mathrm{P}}(1)$ *as* $n \to \infty$, *under* $\mathcal{H}^{(n)}(\boldsymbol{\theta}, \boldsymbol{\Sigma}, \cdot)$.

For each $\mathbf{\Sigma}$ and n, consider the group of transformations $\mathcal{G}_{\mathbf{\Sigma}}^{(n)} = \{\mathcal{G}_g^{(n)}\}$, acting on $(\mathbb{R}^k)^n$, such that $\mathcal{G}_g^{(n)}(\mathbf{X}_1^{(n)}, \ldots, \mathbf{X}_n^{(n)}) := (\boldsymbol{\theta} + g(d_1^{(n)}(\boldsymbol{\theta}, \mathbf{\Sigma}))\mathbf{\Sigma}^{1/2}\mathbf{U}_1^{(n)}(\boldsymbol{\theta}, \mathbf{\Sigma}), \ldots, \boldsymbol{\theta} + g(d_n^{(n)}(\boldsymbol{\theta}, \mathbf{\Sigma}))\mathbf{\Sigma}^{1/2}\mathbf{U}_n^{(n)}(\boldsymbol{\theta}, \mathbf{\Sigma}))$, where $g : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous, monotone increasing and such that $g(0) = 0$ and $\lim_{r\to\infty} g(r) = \infty$. The group $\mathcal{G}_{\mathbf{\Sigma}}^{(n)}$ is a generating group for the submodel $\mathcal{H}^{(n)}(\boldsymbol{\theta}, \mathbf{\Sigma}, \cdot)$. Interdirections clearly are invariant under the action of $\mathcal{G}_{\mathbf{\Sigma}}^{(n)}$, and so are the ranks $R_i^{(n)}(\boldsymbol{\theta}, \mathbf{\Sigma})$. Lemma 2 thus entails asymptotic invariance of the pseudo-Mahalanobis ranks $\hat{R}_i^{(n)}(\boldsymbol{\theta})/(n+1)$.

Another group of interest is the group of affine transformations acting on $(\mathbb{R}^k)^n$, more precisely, the group $\mathcal{G}^{(n)} = \{\mathcal{G}_{\mathbf{M}}^{(n)}\}$, where $|\mathbf{M}| > 0$, and $\mathcal{G}_{\mathbf{M}}^{(n)}(\mathbf{X}_1^{(n)}, \ldots, \mathbf{X}_n^{(n)}) := (\boldsymbol{\theta} + \mathbf{M}(\mathbf{X}_1^{(n)} - \boldsymbol{\theta}), \ldots, \boldsymbol{\theta} + \mathbf{M}(\mathbf{X}_n^{(n)} - \boldsymbol{\theta}))$. This group of affine transformations is a generating group for the submodel $\mathcal{H}^{(n)}(\boldsymbol{\theta}, \cdot, f)$; interdirections and pseudo-Mahalanobis ranks clearly are invariant for this group [see Assumption (A4)].

3. Test statistics and their asymptotic distributions. The testing problem (2) under study inherits from the groups $\mathcal{G}_{\mathbf{\Sigma}}^{(n)}$ and $\mathcal{G}^{(n)}$ a strong invariance structure. Classical invariance arguments in such situations suggest considering invariant testing procedures, based on test statistics which are measurable with respect to maximal invariants. A maximal invariant for $\mathcal{G}_{\mathbf{\Sigma}}^{(n)}$ is

$$\big(\mathbf{U}_1^{(n)}(\boldsymbol{\theta}, \mathbf{\Sigma}), \ldots, \mathbf{U}_n^{(n)}(\boldsymbol{\theta}, \mathbf{\Sigma}), R_1^{(n)}(\boldsymbol{\theta}, \mathbf{\Sigma}), \ldots, R_n^{(n)}(\boldsymbol{\theta}, \mathbf{\Sigma})\big)$$

or equivalently,

$$\big(\mathbf{U}_1^{(n)}(\boldsymbol{\theta}, \mathbf{\Sigma}), \big(\mathbf{U}_i^{(n)}(\boldsymbol{\theta}, \mathbf{\Sigma})\big)'\mathbf{U}_j^{(n)}(\boldsymbol{\theta}, \mathbf{\Sigma}), R_1^{(n)}(\boldsymbol{\theta}, \mathbf{\Sigma}), \ldots, R_n^{(n)}(\boldsymbol{\theta}, \mathbf{\Sigma})\big) \qquad (1 \le i < j \le n),$$

where, however, $\mathbf{U}_1^{(n)}(\boldsymbol{\theta}, \mathbf{\Sigma})$ can be dropped due to the fact [see the comment after Assumption (A1′)] that $\mathbf{\Sigma}^{1/2}$ is only defined up to an arbitrary orthogonal transformation. Now, in practice,

$$\big(\big(\mathbf{U}_i^{(n)}(\boldsymbol{\theta}, \mathbf{\Sigma})\big)'\mathbf{U}_j^{(n)}(\boldsymbol{\theta}, \mathbf{\Sigma}), R_1^{(n)}(\boldsymbol{\theta}, \mathbf{\Sigma}), \ldots, R_n^{(n)}(\boldsymbol{\theta}, \mathbf{\Sigma})\big) \qquad (1 \le i < j \le n) \tag{4}$$

is not a statistic, since $\mathbf{\Sigma}$ is not specified under $\mathcal{H}^{(n)}(\boldsymbol{\theta}, \cdot, \cdot)$, and we propose:

(i) replacing $(\mathbf{U}_i^{(n)}(\boldsymbol{\theta}, \mathbf{\Sigma}))'\mathbf{U}_j^{(n)}(\boldsymbol{\theta}, \mathbf{\Sigma})$ with an estimate based on Randles' interdirections, which are both $\mathcal{G}^{(n)}$- and $\mathcal{G}_{\mathbf{\Sigma}}^{(n)}$-invariant (irrespective of the actual value of $\mathbf{\Sigma}$);

(ii) replacing the ranks $R_i^{(n)}(\boldsymbol{\theta}, \mathbf{\Sigma})$ with the estimated ranks $\hat{R}_i^{(n)}(\boldsymbol{\theta})$. These pseudo-Mahalanobis ranks are strictly affine-invariant, and "almost" $\mathcal{G}_{\mathbf{\Sigma}}^{(n)}$-invariant in the sense that they are asymptotically equivalent to the genuinely invariant ones in (4).

3.1. *A class of test statistics based on interdirections and pseudo-Mahalanobis ranks.* Throughout, let $f_\star : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ satisfy Assumption (A3). The test statistics we are considering are of the form

$$Q_{f_\star}^{(n)}(\boldsymbol{\theta}_0) := \frac{k}{n\mathrm{E}[J_{k,f_\star}^2(U)]} \sum_{i,j=1}^{n} J_{k,f_\star}\left(\frac{\hat{R}_i^{(n)}(\boldsymbol{\theta}_0)}{n+1}\right) J_{k,f_\star}\left(\frac{\hat{R}_j^{(n)}(\boldsymbol{\theta}_0)}{n+1}\right) \cos\left(\pi p_{ij}^{(n)}(\boldsymbol{\theta}_0)\right), \tag{5}$$

where U is uniform over $]0,1[$ and $J_{k,f_\star} := \varphi_{f_\star} \circ \tilde{F}_{\star k}^{-1}$. Such statistics can be considered as multivariate extensions of the traditional linear signed rank statistics; the function $f_\star$ determining the scores will be called the *target* radial density. Since $Q_{f_\star}^{(n)}(\boldsymbol{\theta}_0)$ is measurable with respect to interdirections and the pseudo-Mahalanobis ranks, it is invariant under the group of affine transformations of $\mathbb{R}^k$, hence distribution-free under $\mathcal{H}^{(n)}(\boldsymbol{\theta}_0, \cdot, f)$. Moreover, we shall see that $Q_{f_\star}^{(n)}(\boldsymbol{\theta}_0)$ is asymptotically equivalent to a statistic which is strictly invariant under $\mathcal{G}_{\boldsymbol{\Sigma}}^{(n)}$, so that $Q_{f_\star}^{(n)}(\boldsymbol{\theta}_0)$ is asymptotically distribution-free under $\mathcal{H}^{(n)}(\boldsymbol{\theta}_0, \cdot, \cdot)$.

Let us give a few examples. Letting $f_\star(r) := \exp(-r)$ (double-exponential target density), we obtain Randles' multivariate sign statistic

$$\frac{k}{n} \sum_{i,j=1}^{n} \cos\left(\pi p_{ij}^{(n)}(\boldsymbol{\theta}_0)\right),$$

which, for $k = 1$, reduces to the traditional sign test statistic and, for $k = 2$, to the bivariate Blumen test statistic [Blumen (1958)]; see Randles (1989).

The score functions $J_{k,f_\star}(u) = au$, $u \in]0,1[$, $a > 0$, yield the Wilcoxon type statistic

$$\frac{3k}{n(n+1)^2} \sum_{i,j=1}^{n} \hat{R}_i^{(n)}(\boldsymbol{\theta}_0) \hat{R}_j^{(n)}(\boldsymbol{\theta}_0) \cos\left(\pi p_{ij}^{(n)}(\boldsymbol{\theta}_0)\right), \tag{6}$$

which is asymptotically equivalent to the statistic considered by Peters and Randles (1990). The latter reduces, for $k = 1$, to Wilcoxon's traditional signed rank statistic.

Finally, $f_\star(r) = \exp(-r^2/2)$ characterizes a statistic of the van der Waerden type,

$$\frac{1}{n} \sum_{i,j=1}^{n} \sqrt{\Psi_k^{-1}\left(\frac{\hat{R}_i^{(n)}(\boldsymbol{\theta}_0)}{n+1}\right)} \sqrt{\Psi_k^{-1}\left(\frac{\hat{R}_j^{(n)}(\boldsymbol{\theta}_0)}{n+1}\right)} \cos\left(\pi p_{ij}^{(n)}(\boldsymbol{\theta}_0)\right), \tag{7}$$

where Ψ_k stands for the chi-square distribution function with k degrees of freedom.

3.2. *Asymptotic behavior of statistics based on interdirections and pseudo-Mahalanobis ranks.* We now turn to the asymptotic behavior of $Q^{(n)}_{f_\star}(\boldsymbol{\theta}_0)$ under the null hypotheses considered and contiguous alternatives. The following asymptotic representation is the key to all subsequent results.

LEMMA 3. *Assume that* (A1) *through* (A4) *hold*. *Then*, *under* $\mathcal{H}^{(n)}(\boldsymbol{\theta}_0, \boldsymbol{\Sigma}, f)$,

$$Q^{(n)}_{f_\star}(\boldsymbol{\theta}_0) = \tilde{Q}^{(n)}_{f_\star; f}(\boldsymbol{\theta}_0) + o_{\mathrm{P}}(1),$$

where

$$\tilde{Q}^{(n)}_{f_\star; f}(\boldsymbol{\theta}_0) := \frac{k}{n\, E[J^2_{k,f_\star}(U)]} \sum_{i,j=1}^{n} J_{k,f_\star}\big(\tilde{F}_k\big(d_i^{(n)}(\boldsymbol{\theta}_0, \boldsymbol{\Sigma})\big)\big) J_{k,f_\star}\big(\tilde{F}_k\big(d_j^{(n)}(\boldsymbol{\theta}_0, \boldsymbol{\Sigma})\big)\big)$$
$$\times \big(\mathbf{U}_i^{(n)}(\boldsymbol{\theta}_0, \boldsymbol{\Sigma})\big)' \mathbf{U}_j^{(n)}(\boldsymbol{\theta}_0, \boldsymbol{\Sigma}).$$

Since $\tilde{Q}^{(n)}_{f_\star; f}(\boldsymbol{\theta}_0)$, contrary to $Q^{(n)}_{f_\star}(\boldsymbol{\theta}_0)$, is a quadratic form involving sums of independent summands, its asymptotic distribution is easily obtained. Let $C_k(f_1, f_2) := \int_0^1 J_{k,f_1}(u)\, J_{k,f_2}(u)\, du$, where $J_{k,f_l} := \varphi_{f_l} \circ \tilde{F}^{-1}_{lk}$ $(l = 1, 2)$; for simplicity, we also write $C_k(f)$ instead of $C_k(f, f)$. We then have the following results.

PROPOSITION 3. *Assume that* (A1) *through* (A4) *hold*. *Then*, $Q^{(n)}_{f_\star}(\boldsymbol{\theta}_0)$ *is asymptotically chi-square with k degrees of freedom under* $\mathcal{H}^{(n)}(\boldsymbol{\theta}_0, \cdot, \cdot)$, *and asymptotically noncentral chi-square*, *still with k degrees of freedom but with noncentrality parameter* $\boldsymbol{\tau}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\tau}\, C_k^2(f_\star, f)/kC_k(f_\star)$, *under* $\mathcal{H}^{(n)}(\boldsymbol{\theta}_0 + n^{-1/2}\boldsymbol{\tau}, \boldsymbol{\Sigma}, f)$.

PROPOSITION 4. *The sequence of tests* $\phi^{(n)}_{f_\star}(\boldsymbol{\theta}_0)$ *rejecting the null hypothesis* (2) *whenever* $Q^{(n)}_{f_\star}(\boldsymbol{\theta}_0)$ *exceeds the* $(1-\alpha)$*-quantile* $\chi^2_{k,1-\alpha}$ *of a chi-square distribution with k degrees of freedom*:

(i) *has asymptotic level* α, *and*
(ii) *is locally asymptotically maximin*, *at asymptotic level* α, *for* $\mathcal{H}^{(n)}(\boldsymbol{\theta}_0, \cdot, \cdot)$ *against alternatives of the form* $\bigcup_{\boldsymbol{\theta} \neq \boldsymbol{\theta}_0} \mathcal{H}^{(n)}(\boldsymbol{\theta}, \cdot, f_\star)$.

All proofs are given in Appendix B. In view of the asymptotic representation result of Lemma 3, note that it is sufficient to show that $Q^{(n)}_{f_\star}(\boldsymbol{\theta}_0)$ is asymptotically equivalent, under $\mathcal{H}^{(n)}(\boldsymbol{\theta}_0, \boldsymbol{\Sigma}, f_\star)$, to the parametrically locally asymptotically maximin test statistic

$$Q^{(n)}_{\boldsymbol{\Sigma}, f_\star}(\boldsymbol{\theta}_0) := \boldsymbol{\Delta}^{(n)\prime}_{\boldsymbol{\Sigma}, f_\star}(\boldsymbol{\theta}_0) \boldsymbol{\Gamma}^{-1}_{\boldsymbol{\Sigma}, f_\star} \boldsymbol{\Delta}^{(n)}_{\boldsymbol{\Sigma}, f_\star}(\boldsymbol{\theta}_0).$$

3.3. *Asymptotic relative efficiencies.* We now turn to asymptotic relative efficiencies of the tests $\phi^{(n)}_{f_\star}$ with respect to Hotelling's traditional T^2 test.

PROPOSITION 5. *Assume that Assumptions* (A1′) *through* (A4) *hold*, *and let* $D_k(f) := \mathrm{E}[(\tilde{F}_k^{-1}(U))^2]$. *Then*, *the asymptotic relative efficiency of* $\phi^{(n)}_{f_\star}$ *with respect to Hotelling's test*, *under radial density* f, *is*

$$\mathrm{ARE}_{k,f}(\phi^{(n)}_{f_\star}/T^2) = \frac{D_k(f)}{k^2}\frac{C_k^2(f_\star, f)}{C_k(f_\star)}.$$

These ARE values directly follow as the ratios of the corresponding noncentrality parameters in the asymptotic distributions of $\phi^{(n)}_{f_\star}$ and T^2 under local alternatives. See Appendix B for a detailed proof.

Like in the univariate case [see Chernoff and Savage (1958)], the van der Waerden procedure is uniformly more efficient than the parametric Gaussian one—here, the Hotelling test. More precisely, we establish the following generalization of Chernoff and Savage's classical result.

PROPOSITION 6. *Denote by* $\phi^{(n)}_{vdW}$ *the van der Waerden test*, *based on the test statistic* (7). *For any* f *satisfying Assumptions* (A1′) *and* (A2),

$$\mathrm{ARE}_{k,f}(\phi^{(n)}_{vdW}/T^2) \geq 1,$$

where equality holds if and only if f *is Gaussian*.

The proof relies on an extension of the variational argument used in Chernoff and Savage (1958); see Appendix C.

Proposition 5 also allows for extension of the famous Hodges–Lehmann ".864 result" [Hodges and Lehmann (1956)] by computing, for any dimension k, the lower bound for the asymptotic relative efficiency of the Wilcoxon procedure [namely, the test based on (6)] with respect to Hotelling's test. More precisely, we prove the following in Appendix C.

PROPOSITION 7. *Denote by* $\phi^{(n)}_{W}$ *the Wilcoxon rank-based procedure based on the test statistic* (6). *Then*, *if the infimum is taken over all radial densities* f *satisfying Assumptions* (A1′) *and* (A2),

$$\inf_f \mathrm{ARE}_{k,f}(\phi^{(n)}_{W}/T^2) = \frac{81}{500}\frac{(\sqrt{2k-1}+1)^5}{k^2(\sqrt{2k-1}+5)}. \tag{8}$$

The sequence of lower bounds (8) is monotonically decreasing for $k \geq 2$; as the dimension k tends to infinity, it tends to $81/125 = 0.648$. Some numerical values are presented in Table 1. The reader is referred to the proof of Proposition 7

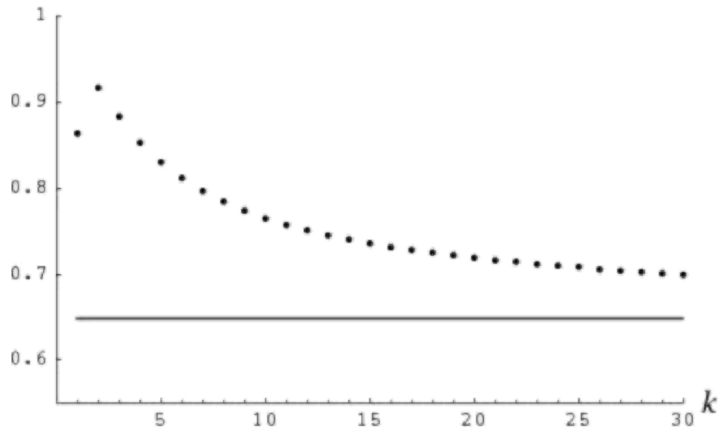

FIG. 2. *Plot of the values of the lower bound* (8) *for the asymptotic relative efficiency of the Wilcoxon rank-based procedure with respect to Hotelling's* T^2 *test, for space dimension* $k = 1, 2, \ldots, 30$. *The horizontal line corresponds to the asymptotic value* 0.648 *of this lower bound as* $k \to \infty$.

in Appendix C for an explicit form of the densities at which this infimum is reached.

In order to evaluate the asymptotic performance of the proposed tests, we now consider the particular case of a multivariate Student density with $\nu_\star$ degrees of freedom. Recall that a k-dimensional random vector $\mathbf{X}$ is multivariate Student with ν degrees of freedom if and only if there exist a vector $\boldsymbol{\theta} \in \mathbb{R}^k$ and a symmetric $k \times k$ positive definite matrix $\boldsymbol{\Sigma}$ such that the density of $\mathbf{X}$ can be written as

$$\frac{\Gamma((k+\nu)/2)}{(\pi\nu)^{k/2}\,\Gamma(\nu/2)}(\det \boldsymbol{\Sigma})^{-1/2} f_\nu(\|\mathbf{x}\|_{\boldsymbol{\theta},\boldsymbol{\Sigma}}),$$

with $f_\nu(r) := (1 + r^2/\nu)^{-(k+\nu)/2}$. Fix $\nu_\star > 2$, and consider the test $\phi^{(n)}_{f_{\nu_\star}}$ associated with the radial density $f_{\nu_\star}$. Since $\varphi_{f_{\nu_\star}}(r) = (k + \nu_\star)r/(\nu_\star + r^2)$, and since the distribution of $\|\mathbf{X}\|^2/k$ under $\mathcal{H}^{(n)}(\mathbf{0}, \mathbf{I}_k, f_{\nu_\star})$ is Fisher–Snedecor with k and $\nu_\star$

TABLE 1
Some numerical values of the lower bound (8) *for the asymptotic relative efficiency of the Wilcoxon rank-based procedure with respect to Hotelling's* T^2 *test, for various values of the space dimension k*

k	**1**	**2**	**3**	**4**	**6**	**10**	**$+\infty$**
$\inf_f \mathrm{ARE}_{k,f}(\phi_W^{(n)}/T^2)$	0.864	0.916	0.883	0.853	0.811	0.765	0.648

TABLE 2
AREs of some $\phi^{(n)}_{f_\star}$ tests with respect to Hotelling, under various k-dimensional Student (3, 4, 6, 8, 10, 15 *and* 20 *degrees of freedom*) *and normal densities, for various values of k*

		Degrees of freedom of the underlying t density							
k	$\mathbf{f}_\star$	**3**	**4**	**6**	**8**	**10**	**15**	**20**	∞
1	t_3	2.000	1.417	1.124	1.025	0.975	0.916	0.890	0.820
	t_6	1.926	1.414	1.167	1.087	1.049	1.005	0.985	0.936
	t_{15}	1.786	1.345	1.143	1.083	1.055	1.026	1.014	0.987
	$\mathcal{N}$	1.639	1.257	1.093	1.048	1.030	1.013	1.007	1.000
2	t_3	2.143	1.488	1.157	1.045	0.987	0.920	0.889	0.807
	t_6	2.067	1.484	1.200	1.107	1.062	1.009	0.986	0.927
	t_{15}	1.910	1.407	1.173	1.102	1.068	1.032	1.018	0.984
	$\mathcal{N}$	1.729	1.301	1.112	1.059	1.037	1.016	1.009	1.000
3	t_3	2.250	1.544	1.186	1.063	1.000	0.926	0.892	0.799
	t_6	2.174	1.540	1.227	1.124	1.073	1.014	0.988	0.919
	t_{15}	2.006	1.458	1.198	1.118	1.080	1.038	1.021	0.981
	$\mathcal{N}$	1.798	1.336	1.128	1.069	1.043	1.019	1.011	1.000
4	t_3	2.333	1.589	1.210	1.079	1.012	0.932	0.896	0.794
	t_6	2.258	1.584	1.250	1.139	1.083	1.019	0.990	0.913
	t_{15}	2.084	1.499	1.220	1.132	1.090	1.044	1.025	0.979
	$\mathcal{N}$	1.853	1.364	1.142	1.077	1.049	1.022	1.012	1.000
6	t_3	2.455	1.657	1.248	1.106	1.033	0.945	0.904	0.788
	t_6	2.382	1.652	1.286	1.163	1.101	1.028	0.995	0.905
	t_{15}	2.202	1.564	1.254	1.155	1.107	1.054	1.031	0.975
	$\mathcal{N}$	1.935	1.408	1.164	1.092	1.059	1.027	1.016	1.000
10	t_3	2.600	1.741	1.299	1.145	1.065	0.968	0.922	0.785
	t_6	2.534	1.736	1.333	1.196	1.126	1.043	1.005	0.896
	t_{15}	2.355	1.649	1.302	1.188	1.132	1.068	1.040	0.969
	$\mathcal{N}$	2.041	1.467	1.195	1.112	1.074	1.035	1.021	1.000

degrees of freedom, the test statistic $Q^{(n)}_{f_{\nu_\star}}(\boldsymbol{\theta}_0)$ is

$$\frac{(k+\nu_\star)(k+\nu_\star+2)}{n}\sum_{i,j=1}^{n}\frac{T_i^{(n)}(\boldsymbol{\theta}_0)}{\nu_\star+(T_i^{(n)}(\boldsymbol{\theta}_0))^2}\frac{T_j^{(n)}(\boldsymbol{\theta}_0)}{\nu_\star+(T_j^{(n)}(\boldsymbol{\theta}_0))^2}\cos\bigl(\pi p_{ij}^{(n)}(\boldsymbol{\theta}_0)\bigr),$$

where, denoting by $G_{k,\nu}$ the Fisher–Snedecor distribution function (k and ν degrees of freedom),

$$T_i^{(n)}(\boldsymbol{\theta}_0) := \sqrt{k\,G^{-1}_{k,\nu_\star}\biggl(\frac{\hat{R}_i^{(n)}(\boldsymbol{\theta}_0)}{n+1}\biggr)}.$$

Table 2 reports the AREs of the tests $\phi^{(n)}_{f_3}$, $\phi^{(n)}_{f_6}$, $\phi^{(n)}_{f_{15}}$, as well as those of the van der Waerden tests $\phi^{(n)}_{vdW}$. All AREs are taken with respect to Hotelling,

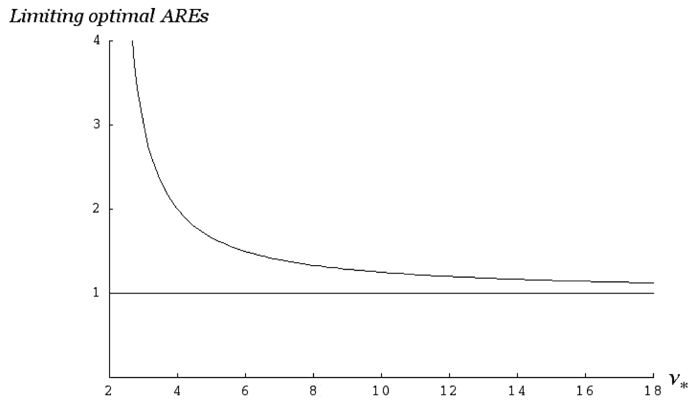

FIG. 3. *Plot of the values of the limiting AREs, as the dimension k of the observation space tends to infinity, of* $\phi^{(n)}_{f_{\nu_\star}}$ *with respect to Hotelling under the corresponding k-dimensional Student, for* $2 < \nu_\star \le 18$; *see* (9).

under k-variate Student densities with various degrees of freedom ν, including the Gaussian density obtained for $\nu = \infty$. This allows for an investigation of the relative performances of the tests under study as a function of the tails of radial densities.

Inspection of Table 2 shows that, as expected, $\phi^{(n)}_{f_{\nu_\star}}$ (respectively, $\phi^{(n)}_{vdW}$) performs best when the underlying density is Student with $\nu_\star$ degrees of freedom (respectively, normal). All tests however exhibit rather good performance, particularly under heavy tailed densities. Note that the van der Waerden test indeed performs uniformly better than Hotelling, which provides an empirical confirmation of Proposition 6.

Since $D_k(f_{\nu_\star}) = k\nu_\star/(\nu_\star - 2)$, we easily obtain that

$$\text{ARE}_{k, f_{\nu_\star}}[\phi^{(n)}_{f_{\nu_\star}}/T^2] = \frac{(k+\nu_\star)\nu_\star}{(k+\nu_\star+2)(\nu_\star - 2)}, \tag{9}$$

a quantity that increases with k, and tends to $\nu_\star/(\nu_\star - 2)$ as $k \to \infty$. The advantage of $\phi^{(n)}_{f_{\nu_\star}}$ over Hotelling thus increases with the dimension k of the observations. Table 3 presents some of these limiting ARE values.

TABLE 3
Limiting AREs, as the dimension k of the observation space tends to infinity, of some $\phi^{(n)}_{f_{\nu_\star}}$ *tests with respect to Hotelling, under the corresponding k-dimensional Student and normal densities*; *see* (9)

Degrees of freedom of the underlying t density											
3	4	5	6	7	8	10	12	15	20	50	∞
3.000	2.000	1.667	1.500	1.400	1.333	1.250	1.200	1.154	1.111	1.042	1.000
Limiting ARE values											

TABLE 4

AREs with respect to Hotelling of (S) *Randles' multivariate sign test, Oja's spatial sign test and Oja's affine-invariant sign test,* (W) *of Peters and Randles' Wilcoxon type multivariate signed rank test,* (SR) *of Oja's spatial signed rank test, and* (OR) *Oja's affine-invariant signed rank test, under various k-variate t and normal densities,* $k = 1, 2, 3, 4, 6, 10$

		Degrees of freedom of the underlying t density							
k	**test**	**3**	**4**	**6**	**8**	**10**	**15**	**20**	∞
1	S	1.621	1.125	0.879	0.798	0.757	0.710	0.690	0.637
	W	1.900	1.401	1.164	1.089	1.054	1.014	0.997	0.955
	SR	1.900	1.401	1.164	1.089	1.054	1.014	0.997	0.955
	OR	1.900	1.401	1.164	1.089	1.054	1.014	0.997	0.955
2	S	2.000	1.388	1.084	0.984	0.934	0.877	0.851	0.785
	W	1.748	1.317	1.123	1.066	1.041	1.015	1.005	0.985
	SR	1.953	1.435	1.187	1.108	1.071	1.029	1.011	0.967
	OR	2.026	1.469	1.196	1.107	1.064	1.014	0.992	0.937
3	S	2.162	1.500	1.172	1.063	1.009	0.947	0.920	0.849
	W	1.621	1.233	1.064	1.019	1.000	0.983	0.978	0.975
	SR	1.994	1.453	1.200	1.119	1.081	1.038	1.019	0.973
	OR	2.112	1.515	1.221	1.124	1.076	1.021	0.997	0.934
4	S	2.250	1.561	1.220	1.107	1.051	0.986	0.958	0.884
	W	1.533	1.171	1.018	0.979	0.964	0.954	0.952	0.961
	SR	2.018	1.467	1.208	1.127	1.087	1.044	1.025	0.978
	OR	2.173	1.550	1.241	1.139	1.088	1.030	1.004	0.937
6	S	2.344	1.626	1.271	1.153	1.094	1.027	0.997	0.920
	W	1.422	1.090	0.953	0.921	0.911	0.908	0.911	0.938
	SR	2.050	1.484	1.219	1.136	1.095	1.051	1.031	0.984
	OR	2.256	1.598	1.270	1.162	1.108	1.045	1.018	0.947
10	S	2.422	1.681	1.313	1.192	1.131	1.062	1.031	0.951
	W	1.315	1.007	0.882	0.855	0.848	0.851	0.857	0.907
	SR	2.093	1.503	1.229	1.144	1.103	1.058	1.038	0.989
	OR	2.346	1.650	1.304	1.189	1.132	1.066	1.037	0.961

Table 4 provides the asymptotic relative efficiencies, still with respect to Hotelling, and under the same densities as in Tables 2 and 3, of some of the tests proposed in the literature. These ARE figures allow for comparing our tests with their competitors. The following tests have been considered:

(a) Randles' (1989) multivariate sign test (S1),
(b) Peters and Randles' (1990) generalized Wilcoxon test (W),
(c) the spatial sign test [Möttönen and Oja (1995)] (S2),
(d) the spatial signed rank test [Möttönen and Oja (1995)] (SR),
(e) the affine-invariant sign test [Hettmansperger et al. (1994)] (S3) and
(f) the affine-invariant signed rank test [Hettmansperger et al. (1997)] (OR), both based on the so-called Oja median.

ARE values for (S1) and (W) are computed from the result in Proposition 5; for (S1), the following closed-form expression is obtained:

$$\text{ARE}_{k,f_\nu}[S1/T^2] = \frac{4}{k(\nu-2)}\left[\frac{\Gamma((k+1)/2)}{\Gamma(k/2)}\frac{\Gamma((\nu+1)/2)}{\Gamma(\nu/2)}\right]^2. \tag{10}$$

Since (S2) and (S3) are asymptotically equivalent, under spherical symmetry, with Randles' multivariate sign test (S1), their AREs coincide with (10); note that this entails that Oja's sign tests are locally asymptotically optimal under spherical symmetry and double-exponential radial density. The same optimality property still holds for (S3)—but not for (S2), which is only rotationally invariant—under elliptical symmetry (still, with double-exponential radial density). The reader is referred to Möttönen et al. (1997, 1998) for the derivation of ARE values for (SR) and (OR).

4. Finite sample performance.

4.1. *Simulations.* The following Monte Carlo experiment was conducted in order to investigate the finite-sample behavior of the tests proposed in Section 3 for $k = 2$ and (without loss of generality) $\boldsymbol{\theta}_0 = (0, 0)$: $N = 2{,}500$ independent samples $(\mathbf{X}_1, \ldots, \mathbf{X}_{200})$ of size $n = 200$ have been generated from bivariate standard Student densities with 3, 6 and 15 degrees of freedom, and from the bivariate standard normal distribution. The simulation of bivariate Student variables $\mathbf{X}_i$ was based on the fact that (for ν degrees of freedom) $\mathbf{X}_i =_{\mathrm{d}} \mathbf{Z}_i/\sqrt{Y_i/\nu}$, where $\mathbf{Z}_i \sim \mathcal{N}_2(\mathbf{0}, \mathbf{I}_2)$ and $Y_i \sim \chi^2_\nu$ are independent. For each replication, the following seven tests were performed at nominal probability level $\alpha = 5\%$: the Hotelling test, $\phi^{(n)}_{f_3}, \phi^{(n)}_{f_6}, \phi^{(n)}_{f_{15}}, \phi^{(n)}_{vdW}$, Randles' sign test (S), and the signed rank test of Peters and Randles (W). Tyler's estimator of scatter was used whenever pseudo-Mahalanobis ranks had to be computed. The estimator was obtained via the iterative scheme described in Randles (2000). Iterations were stopped as soon as the Frobenius distance between the left- and right-hand sides of (3) fell below 10^{-6}.

Rejection frequencies, estimating the corresponding size and powers, were recorded at four values of $\boldsymbol{\theta}$, of the form $m\boldsymbol{\Delta}$, with $\boldsymbol{\Delta} = (0.06, 0.06)$ and $m = 0, 1, 2, 3$; they are reported in Table 5. Note that the standard errors for such estimates are (for 2,500 replications) 0.0044, 0.0080 and 0.0100 for estimated probabilities (size or power) of the order of $p = 0.05$ (equivalently, 0.95), $p = 0.20$ (equivalently, 0.80), and $p = 0.50$, respectively.

All tests apparently satisfy the 5% probability level constraint (a confidence interval with confidence level 95% has approximate half-length 0.01); some of them (such as Hotelling's) seem to be significantly biased. Power rankings are essentially consistent with the ARE rankings given in Tables 2 and 4. For instance, under Gaussian densities, the powers of the $\phi_{f_{\nu_\star}}$ tests are increasing with $\nu_\star$, as expected, whereas the asymptotic optimality of $\phi^{(n)}_{f_{\nu_\star}}$ under the Student distribution with $\nu_\star$ degrees of freedom is confirmed.

TABLE 5

Estimated sizes and powers of the Hotelling test, the $\phi_{f_3}^{(n)}$, $\phi_{f_6}^{(n)}$, $\phi_{f_{15}}^{(n)}$ and $\phi_{vdW}^{(n)}$ tests, Randles' sign test (S), *and the signed rank test of Peters and Randles* (W), *under various values of the shift and various densities; simulations are based on* 2,500 *replications*

		Shift					**Shift**			
Test	**Density**	**0**	**Δ**	**2Δ**	**3Δ**	**Density**	**0**	**Δ**	**2Δ**	**3Δ**
T^2	$\mathcal{N}$	0.0572	0.1844	0.5868	0.9108	t_6	0.0628	0.1452	0.4288	0.7680
ϕ_{vdW}		0.0544	0.1788	0.5872	0.9076		0.0572	0.1484	0.4668	0.7984
$\phi_{f_{15}}$		0.0548	0.1808	0.5816	0.9020		0.0568	0.1588	0.4976	0.8252
ϕ_{f_6}		0.0576	0.1740	0.5608	0.8888		0.0536	0.1576	0.5016	0.8328
ϕ_{f_3}		0.0552	0.1588	0.5084	0.8404		0.0512	0.1568	0.4868	0.8140
S		0.0504	0.1476	0.4916	0.8228		0.0504	0.1456	0.4568	0.7872
W		0.0588	0.1816	0.5852	0.9032		0.0592	0.1520	0.4744	0.8040
T^2	t_{15}	0.0544	0.1644	0.5212	0.8612	t_3	0.0456	0.0976	0.2656	0.5252
ϕ_{vdW}		0.0524	0.1628	0.5284	0.8624		0.0524	0.1264	0.3712	0.6876
$\phi_{f_{15}}$		0.0560	0.1724	0.5456	0.8704		0.0540	0.1340	0.4096	0.7316
ϕ_{f_6}		0.0552	0.1688	0.5420	0.8668		0.0512	0.1436	0.4408	0.7628
ϕ_{f_3}		0.0528	0.1584	0.5056	0.8364		0.0500	0.1488	0.4568	0.7788
S		0.0504	0.1476	0.4780	0.8108		0.0504	0.1384	0.4176	0.7564
W		0.0588	0.1688	0.5332	0.8656		0.0552	0.1316	0.3776	0.6896

4.2. *A numerical example.* We are treating an example of Brown et al. (1992), also considered by Möttönen and Oja (1995) and by Oja (1999). The data consist of a sample of differences of refraction measures for the left (first component) and right (second component) eyes, respectively, between $n = 10$ fathers and their sons. The null hypothesis of interest is the absence of a difference between fathers and sons, yielding the testing problem (2) with $\boldsymbol{\theta}_0 = (0, 0)$.

The Hotelling statistic for these data takes value 4.751, with p-value 0.183 (corresponding to a Fisher–Snedecor distribution with 2 and 8 degrees of freedom). If the empirical variance-covariance matrix is used, the test statistics $Q_{f_3}^{(n)}$, $Q_{f_6}^{(n)}$, $Q_{f_{15}}^{(n)}$ and $Q_{vdW}^{(n)}$ take values 3.718, 3.627, 3.282 and 2.883, with (asymptotic approximations) p-values 0.156, 0.163, 0.194 and 0.237, respectively. If Tyler's estimator of multivariate scatter is used, the resulting test statistics take values 3.389, 3.018, 2.599 and 2.216, with (asymptotic) p-values 0.184, 0.221, 0.273 and 0.330, respectively.

TABLE 6

Differences of refraction measures between 10 *fathers and their sons for left and right eyes, respectively*

Left eyes	1.12	−1.75	−2.50	1.50	0.25	−3.00	−1.50	1.50	−1.62	−1.00
Right eyes	1.75	−2.37	−2.75	1.25	−0.13	−3.25	−2.25	−0.50	−0.62	−1.75

These p-values are to be compared with those achieved by the statistics (S1) of Randles (1989) (2.894; p-value 0.235), (W) of Peters and Randles (1990) (3.411 and 2.362, with p-values 0.182 and 0.307, for the empirical variance-covariance estimator and the Tyler estimator of scatter, respectively), and by Oja's (S2), (SR), (S3) and (OR) test statistics (3.405, 3.587, 3.241 and 3.849; p-values 0.182, 0.166, 0.198 and 0.146, respectively).

5. Final comments. While enjoying the same attractive invariance properties (with respect to affine transformations and the group of radial order-preserving transformations) as the multivariate nonparametric methods developed by Hettmansperger, Oja, Randles and their coauthors, the tests we are proposing also exhibit local asymptotic optimality a la Le Cam. We show that, in their van der Waerden version, they are uniformly better than Hotelling's classical test. Under the simpler Wilcoxon form, we prove that their asymptotic relative efficiency, irrespective of the dimension of the underlying observation space, never falls below 0.648.

All tests have been described under asymptotic form, with critical values derived from asymptotic distributions. It is worth mentioning that, in practice, the same tests can be based on permutational critical values. Such values can be obtained from sampling the 2^n possible values $s_1(\mathbf{X}_1^{(n)} - \boldsymbol{\theta}_0), \dots, s_n(\mathbf{X}_n^{(n)} - \boldsymbol{\theta}_0)$ ($\mathbf{s} \in \{-1, 1\}^n$), which are equally probable under the null. A pleasant feature of this approach is that the (at most) 2^n corresponding possible values of the test statistics can be based on a unique evaluation of the interdirections and the (pseudo-)Mahalanobis ranks. Indeed, denoting by $p_{ij}^{(n)}(\mathbf{s})$ the interdirection associated with the pair $(s_i(\mathbf{X}_i^{(n)} - \boldsymbol{\theta}_0), s_j(\mathbf{X}_j^{(n)} - \boldsymbol{\theta}_0))$ in the residual n-tuple $s_1(\mathbf{X}_1^{(n)} - \boldsymbol{\theta}_0), \dots, s_n(\mathbf{X}_n^{(n)} - \boldsymbol{\theta}_0)$, it can be checked [see Randles (1989)] that $\cos(\pi\, p_{ij}^{(n)}(\mathbf{s})) = s_i s_j \cos(\pi p_{ij}^{(n)})$. It follows that the test statistic $Q_{f_\star}^{(n)}(\mathbf{s})$ computed from $s_1(\mathbf{X}_1^{(n)} - \boldsymbol{\theta}_0), \dots, s_n(\mathbf{X}_n^{(n)} - \boldsymbol{\theta}_0)$ is

$$\frac{k}{n\mathrm{E}[J_{k,f_\star}^2(U)]} \sum_{i,j=1}^{n} s_i J_{k,f_\star}\biggl(\frac{\hat{R}_i^{(n)}(\boldsymbol{\theta}_0)}{n+1}\biggr) s_j J_{k,f_\star}\biggl(\frac{\hat{R}_j^{(n)}(\boldsymbol{\theta}_0)}{n+1}\biggr) \cos\bigl(\pi p_{ij}^{(n)}(\boldsymbol{\theta}_0)\bigr).$$

The situation is much less comfortable with the permutational versions of Hotelling's test, or Oja's (S2), (S3), (SR) and (OR) procedures.

On the other hand, the methods we are proposing are limited to elliptically symmetric models, whereas the Oja methods remain valid under weaker central symmetry assumptions $(\mathbf{X}_i - \boldsymbol{\theta}) =_{\mathrm{d}} -(\mathbf{X}_i - \boldsymbol{\theta})$, $i = 1, \dots, n$. The behavior under nonelliptical conditions of interdirections remains a largely open problem.

APPENDIX A

Quadratic mean differentiability of $\mathbf{x} \mapsto f^{1/2}(\|\mathbf{x}\|)$. In this section, we essentially prove Proposition 1. Note that h belongs to $L^2(\mathbb{R}_0^+, \mu_{k-1})$ if and only if $\mathbf{x} \mapsto g(\mathbf{x}) := h(\|\mathbf{x}\|)$ belongs to $L^2(\mathbb{R}^k)$. The proof of Proposition 1 relies on the following lemma, which establishes that a function $\mathbf{x} \mapsto g(\mathbf{x})$ is differentiable in quadratic mean over $\mathbb{R}^k$ if and only if it admits a derivative in the sense of distributions, provided that the gradient in the sense of distributions $\boldsymbol{\partial} g$ be square-summable over $\mathbb{R}^k$.

Denote by ∇g, $\boldsymbol{\partial} g$ and $\mathbf{D}g$ the gradients of g in the classical sense, in the sense of distributions, and in quadratic mean, respectively. The following lemma is then a particular case of a result in Schwartz (1973), pages 186–188.

LEMMA 4. *Let $g : \mathbb{R}^k \to \mathbb{R}$. Then, g is differentiable in quadratic mean over $\mathbb{R}^k$ if and only if $g \in W^{1,2}(\mathbb{R}^k)$. Moreover, $\boldsymbol{\partial} g = \mathbf{D}g$ in $L^2(\mathbb{R}^k)$.*

This result enables us to prove Proposition 1.

PROOF OF PROPOSITION 1. (i) We first show that $h \in W^{1,2}(\mathbb{R}_0^+, \mu_{k-1})$ is a necessary condition for $g : \mathbf{x} \mapsto h(\|\mathbf{x}\|)$ being quadratic mean differentiable. We already pointed out that h had to be in $L^2(\mathbb{R}_0^+, \mu_{k-1})$. On the other hand, because of symmetry, the derivative of g in the sense of distributions is clearly of the form $\boldsymbol{\partial} g(\mathbf{x}) = s(\|\mathbf{x}\|)\mathbf{x}/\|\mathbf{x}\|$ (this could be made precise via a regularization argument). Since this derivative is assumed to be square-integrable (see Lemma 4), $s \in L^2(\mathbb{R}_0^+, \mu_{k-1})$. We still have to show that s is the weak derivative of h in $L^2(\mathbb{R}_0^+, \mu_{k-1})$. To do so, associate with any function $\psi \in C_0^\infty(\mathbb{R}_0^+)$ the function $\bar{\psi}$ defined on $\mathbb{R}^k$ by $\bar{\psi}(\mathbf{x}) := \psi(\|\mathbf{x}\|)m(\mathbf{x}/\|\mathbf{x}\|)$, where m is some real-valued function defined on the unit sphere $\mathcal{S}^{k-1}$ that satisfies $c_m := \int_{\mathcal{S}^{k-1}} m(\mathbf{u})\, d\sigma(\mathbf{u}) \neq 0$, where σ stands for the uniform measure over the sphere $\mathcal{S}^{k-1}$, equipped with its Borel sigma-field. We then have

$$\int \left(\sum_{i=1}^k (\partial_i g)(\mathbf{x}) \frac{x_i}{\|\mathbf{x}\|} \right) \bar{\psi}(\mathbf{x})\, d\mathbf{x} = c_m \left(\int_0^\infty s(r)\psi(r) r^{k-1}\, dr \right)$$

and

$$\begin{aligned}
&\int \left(\sum_{i=1}^k (\partial_i g)(\mathbf{x}) \frac{x_i}{\|\mathbf{x}\|} \right) \bar{\psi}(\mathbf{x})\, d\mathbf{x} \\
&\qquad = -\sum_{i=1}^k \int g(\mathbf{x}) \partial_i \left(\frac{x_i}{\|\mathbf{x}\|} \bar{\psi}(\mathbf{x}) \right) d\mathbf{x} \\
&\qquad = -\int g(\mathbf{x}) \left[\frac{k-1}{\|\mathbf{x}\|} \bar{\psi}(\mathbf{x}) + \psi'(\|\mathbf{x}\|) m(\mathbf{x}/\|\mathbf{x}\|) \right] d\mathbf{x} \\
&\qquad = -c_m \left(\int_0^\infty h(r) \left[\frac{k-1}{r} \psi(r) + \psi'(r) \right] r^{k-1}\, dr \right) \\
&\qquad = -c_m \left(\int_0^\infty h(r) [r^{k-1}\psi(r)]'\, dr \right).
\end{aligned}$$

Since this holds for any function $\psi \in C_0^\infty(\mathbb{R}_0^+)$, s is indeed the weak derivative of h in $L^2(\mathbb{R}_0^+, \mu_{k-1})$.

(ii) The proof that $h \in W^{1,2}(\mathbb{R}_0^+, \mu_{k-1})$ is also sufficient for $g : \mathbf{x} \mapsto h(\|\mathbf{x}\|)$ to be quadratic mean differentiable follows from a regularization argument. Define $h_n := h * \rho_n$ ($n \in \mathbb{N}_0$), where $*$ denotes convolution, $\rho_n(x) := n\rho(nx)$, $\rho \in C_0^\infty$ is even and integrates up to one. Then, $h_n \to h$ in $L^2(\mathbb{R}_0^+, \mu_{k-1})$. Moreover, $h_n' = (\rho_n * h)' = \rho_n * h' \to h'$ in $L^2(\mathbb{R}_0^+, \mu_{k-1})$. Put $g_n(\cdot) := h_n(\|\cdot\|)$. Since h_n belongs to $L^2(\mathbb{R}_0^+, \mu_{k-1})$, g_n also belongs to $L^2(\mathbb{R}^k)$. Furthermore,

$$\|g_n - g\|_{L^2(\mathbb{R}^k)} = \|h_n - h\|_{L^2(\mathbb{R}_0^+, \mu_{k-1})} = o(1),$$

as $n \to \infty$. On the other hand, g_n is almost everywhere differentiable in the classical sense, with a.e. gradient $\nabla g_n(\mathbf{x}) = h_n'(\|\mathbf{x}\|)\,\mathbf{x}/\|\mathbf{x}\|$. Since $h_n' \in L^2(\mathbb{R}_0^+, \mu_{k-1})$, ∇g_n belongs to $L^2(\mathbb{R}^k)$. It follows that g_n is differentiable in the sense of distributions, with gradient $\boldsymbol{\partial} g_n(\mathbf{x}) = h_n'(\|\mathbf{x}\|)\,\mathbf{x}/\|\mathbf{x}\|$. Finally, defining $\mathbf{T} \in L^2(\mathbb{R}^k)$ as $\mathbf{T}(\mathbf{x}) := h'(\|\mathbf{x}\|)\,\mathbf{x}/\|\mathbf{x}\|$ (h' here denotes the *weak* derivative of h),

$$\|\boldsymbol{\partial} g_n - \mathbf{T}\|_{L^2(\mathbb{R}^k)} = \|h_n' - h'\|_{L^2(\mathbb{R}_0^+, \mu_{k-1})} = o(1),$$

as $n \to \infty$. From this we may conclude that (g_n) is a Cauchy sequence in $W^{1,2}(\mathbb{R}^k)$, hence that $g_n \to g$ in $W^{1,2}(\mathbb{R}^k)$, and $\boldsymbol{\partial} g = \mathbf{T}$. As a consequence, g is differentiable in the sense of distributions, with square summable gradient, which, in view of Lemma 4, implies that g is differentiable in quadratic mean. In addition, $\mathbf{D}g$ can be chosen to be $\boldsymbol{\partial} g = \mathbf{T}$, which completes the proof. $\square$

APPENDIX B

Proofs of Section 3. The main task here consists in proving Lemma 3.

PROOF OF LEMMA 3. Without loss of generality, we may assume that $\boldsymbol{\Sigma} = \mathbf{I}_k$. Decompose $Q_{f_\star}^{(n)}(\boldsymbol{\theta}_0) - \tilde{Q}_{f_\star; f}^{(n)}(\boldsymbol{\theta}_0)$ into $\frac{k}{\mathrm{E}[J_{k,f_\star}^2(U)]}\,(T_1^{(n)} + T_2^{(n)})$, where

$$T_1^{(n)} := \frac{1}{n} \sum_{i,j=1}^{n} J_{k,f_\star}\big(\hat{R}_i^{(n)}(\boldsymbol{\theta}_0)/(n+1)\big) J_{k,f_\star}\big(\hat{R}_j^{(n)}(\boldsymbol{\theta}_0)/(n+1)\big) \times \big(\cos\big(\pi p_{ij}^{(n)}(\boldsymbol{\theta}_0)\big) - \big(\mathbf{U}_i^{(n)}(\boldsymbol{\theta}_0, \mathbf{I}_k)\big)' \mathbf{U}_j^{(n)}(\boldsymbol{\theta}_0, \mathbf{I}_k)\big)$$

and

$$T_2^{(n)} := \frac{1}{n} \sum_{i,j=1}^{n} \Big(J_{k,f_\star}\big(\hat{R}_i^{(n)}(\boldsymbol{\theta}_0, \mathbf{I}_k)/(n+1)\big) J_{k,f_\star}\big(\hat{R}_j^{(n)}(\boldsymbol{\theta}_0, \mathbf{I}_k)/(n+1)\big) - J_{k,f_\star}\big(\tilde{F}_k\big(d_i^{(n)}(\boldsymbol{\theta}_0, \mathbf{I}_k)\big)\big) J_{k,f_\star}\big(\tilde{F}_k\big(d_j^{(n)}(\boldsymbol{\theta}_0, \mathbf{I}_k)\big)\big)\Big) \times \big(\mathbf{U}_i^{(n)}(\boldsymbol{\theta}_0, \mathbf{I}_k)\big)' \mathbf{U}_j^{(n)}(\boldsymbol{\theta}_0, \mathbf{I}_k).$$

Let us show that, under $\mathcal{H}^{(n)}(\boldsymbol{\theta}_0, \mathbf{I}_k, f)$ [throughout this proof, all convergences and mathematical expectations are taken under $\mathcal{H}^{(n)}(\boldsymbol{\theta}_0, \mathbf{I}_k, f)$], there exists $s > 0$ such that $T_1^{(n)}, T_2^{(n)} \xrightarrow{L^s} 0$ as $n \to \infty$. Slutzky's classical argument then concludes the proof.

Let us start with $T_2^{(n)}$. Defining

$$\mathbf{T}^{(n)}_{f_\star;f}(\boldsymbol{\theta}_0) := n^{-1/2} \sum_{i=1}^{n} J_{k,f_\star}\big(\tilde{F}_k\big(d_i^{(n)}(\boldsymbol{\theta}_0, \mathbf{I}_k)\big)\big)\mathbf{U}_i^{(n)}(\boldsymbol{\theta}_0, \mathbf{I}_k), \tag{11}$$

$$\mathbf{S}^{(n)}_{f_\star}(\boldsymbol{\theta}_0) := n^{-1/2} \sum_{i=1}^{n} J_{k,f_\star}\big(R_i^{(n)}(\boldsymbol{\theta}_0, \mathbf{I}_k)/(n+1)\big)\mathbf{U}_i^{(n)}(\boldsymbol{\theta}_0, \mathbf{I}_k)$$

and

$$\hat{\mathbf{S}}^{(n)}_{f_\star}(\boldsymbol{\theta}_0) := n^{-1/2} \sum_{i=1}^{n} J_{k,f_\star}\big(\hat{R}_i^{(n)}(\boldsymbol{\theta}_0)/(n+1)\big)\mathbf{U}_i^{(n)}(\boldsymbol{\theta}_0, \mathbf{I}_k),$$

note that

$$\begin{aligned}&\big\|\mathbf{T}^{(n)}_{f_\star;f}(\boldsymbol{\theta}_0) - \mathbf{S}^{(n)}_{f_\star}(\boldsymbol{\theta}_0)\big\|^2_{L^2}\\ &\qquad = \sum_{i=1}^{n} (c_i^{(n)})^2 \mathrm{E}\Big[\big(J_{k,f_\star}\big(R_i^{(n)}(\boldsymbol{\theta}_0, \mathbf{I}_k)/(n+1)\big) - J_{k,f_\star}\big(\tilde{F}_k(d_i^{(n)}(\boldsymbol{\theta}_0, \mathbf{I}_k))\big)\big)^2\Big],\end{aligned}$$

where $c_i^{(n)} = n^{-1/2}$ for all $i = 1, \ldots, n$. Hájek's classical projection result thus implies that

$$\big\|\mathbf{T}^{(n)}_{f_\star;f}(\boldsymbol{\theta}_0) - \mathbf{S}^{(n)}_{f_\star}(\boldsymbol{\theta}_0)\big\|^2_{L^2} = o(1)$$

as $n \to \infty$. Incidentally, the same result also implies that, for all $i = 1, \ldots, n$,

$$\mathrm{E}\Big[\big(J_{k,f_\star}\big(R_i^{(n)}(\boldsymbol{\theta}_0, \mathbf{I}_k)/(n+1)\big) - J_{k,f_\star}\big(\tilde{F}_k(d_i^{(n)}(\boldsymbol{\theta}_0, \mathbf{I}_k))\big)\big)^2\Big] = o(1) \tag{12}$$

as $n \to \infty$.

Noting that $\mathrm{E}[G(\mathbf{X}_1^{(n)} - \boldsymbol{\theta}_0, \ldots, \mathbf{X}_n^{(n)} - \boldsymbol{\theta}_0)(\mathbf{U}_i^{(n)}(\boldsymbol{\theta}_0, \mathbf{I}_k))'\mathbf{U}_j^{(n)}(\boldsymbol{\theta}_0, \mathbf{I}_k)] = 0$ for $i \neq j$, for any function G that is even in all its arguments (and for which the expectation exists), we obtain similarly

$$\begin{aligned}&\big\|\mathbf{S}^{(n)}_{f_\star}(\boldsymbol{\theta}_0) - \hat{\mathbf{S}}^{(n)}_{f_\star}(\boldsymbol{\theta}_0)\big\|^2_{L^2}\\ &\qquad = \frac{1}{n} \sum_{i=1}^{n} \mathrm{E}\Big[\big(J_{k,f_\star}\big(R_i^{(n)}(\boldsymbol{\theta}_0, \mathbf{I}_k)/(n+1)\big) - J_{k,f_\star}\big(\hat{R}_i^{(n)}(\boldsymbol{\theta}_0)/(n+1)\big)\big)^2\Big],\end{aligned}$$

which is $o(1)$, if

$$J_{k,f_\star}\big(\hat{R}_1^{(n)}(\boldsymbol{\theta}_0)/(n+1)\big) - J_{k,f_\star}\big(R_1^{(n)}(\boldsymbol{\theta}_0, \mathbf{I}_k)/(n+1)\big) \xrightarrow{L^2} 0 \tag{13}$$

as $n \to \infty$. Lemma 2 establishes the same convergence as in (13), but in probability. We have seen above that $J_{k,f_\star}(R_1^{(n)}(\boldsymbol{\theta}_0, \mathbf{I}_k)/(n+1)) - J_{k,f_\star}(\tilde{F}_k(d_1^{(n)}(\boldsymbol{\theta}_0)))$ tends to zero in quadratic mean, so that $\mathrm{E}[(J_{k,f_\star}(R_1^{(n)}(\boldsymbol{\theta}_0, \mathbf{I}_k)/(n+1)))^2]$ is uniformly integrable. In view of Assumption (A4), the same conclusion holds for $\mathrm{E}[(J_{k,f_\star}(\hat{R}_1^{(n)}(\boldsymbol{\theta}_0, \mathbf{I}_k)/(n+1)))^2]$; (13) follows. Consequently, $\mathbf{S}_{f_\star}^{(n)}(\boldsymbol{\theta}_0) - \hat{\mathbf{S}}_{f_\star}^{(n)}(\boldsymbol{\theta}_0)$, and therefore $\mathbf{T}_{f_\star;f}^{(n)}(\boldsymbol{\theta}_0) - \hat{\mathbf{S}}_{f_\star}^{(n)}(\boldsymbol{\theta}_0)$, vanish in quadratic mean as $n \to \infty$. On the other hand, the sequence $\|\mathbf{T}_{f_\star;f}^{(n)}(\boldsymbol{\theta}_0)\|_{L^2}$ is clearly bounded, so that $\|\hat{\mathbf{S}}_{f_\star}^{(n)}(\boldsymbol{\theta}_0)\|_{L^2}$ also is. Finally, in view of Cauchy–Schwarz,

$$\begin{aligned}
\|T_2^{(n)}\|_{L^1} &= \big\|\big(\hat{\mathbf{S}}_{f_\star}^{(n)}(\boldsymbol{\theta}_0)\big)'\hat{\mathbf{S}}_{f_\star}^{(n)}(\boldsymbol{\theta}_0) - \big(\mathbf{T}_{f_\star;f}^{(n)}(\boldsymbol{\theta}_0)\big)'\mathbf{T}_{f_\star;f}^{(n)}(\boldsymbol{\theta}_0)\big\|_{L^1} \\
&\le \big\|\hat{\mathbf{S}}_{f_\star}^{(n)}(\boldsymbol{\theta}_0) + \mathbf{T}_{f_\star;f}^{(n)}(\boldsymbol{\theta}_0)\big\|_{L^2}\big\|\hat{\mathbf{S}}_{f_\star}^{(n)}(\boldsymbol{\theta}_0) - \mathbf{T}_{f_\star;f}^{(n)}(\boldsymbol{\theta}_0)\big\|_{L^2} \\
&= o(1) \qquad \text{as } n \to \infty.
\end{aligned}$$

It remains to show that $\|T_1^{(n)}\|_{L^2} \to 0$ as $n \to \infty$. Letting

$$B_i^{(n)} := J_{k,f_\star}\bigg(\frac{\hat{R}_i^{(n)}(\boldsymbol{\theta}_0)}{n+1}\bigg)$$

and

$$C_{ij}^{(n)} := \cos\big(\pi p_{ij}^{(n)}(\boldsymbol{\theta}_0)\big) - \big(\mathbf{U}_i^{(n)}(\boldsymbol{\theta}_0, \mathbf{I}_k)\big)'\mathbf{U}_j^{(n)}(\boldsymbol{\theta}_0, \mathbf{I}_k),$$

and taking into account the symmetry properties of interdirections, we easily obtain that

$$\begin{aligned}
\|T_1^{(n)}\|_{L^2}^2 &= \frac{1}{n^2}\mathrm{E}\Bigg[\Bigg(\sum_{i,j=1}^{n} B_i^{(n)}B_j^{(n)}C_{ij}^{(n)}\Bigg)^2\Bigg] = \frac{2(n-1)}{n}\mathrm{E}\big[(B_1^{(n)}B_2^{(n)}C_{12}^{(n)})^2\big] \\
&\le \frac{2(n-1)}{n}\big(\mathrm{E}\big[|B_1^{(n)}B_2^{(n)}|^{2+\delta}\big]\big)^{2/(2+\delta)}\big(\mathrm{E}\big[|C_{12}^{(n)}|^{2(2+\delta)/\delta}\big]\big)^{\delta/(2+\delta)},
\end{aligned}$$

where $\delta > 0$ is as in Assumption (A3) (the last inequality above results from Hölder's inequality). Now, Lemma 1 and the boundedness of $C_{12}^{(n)}$ yield that $\mathrm{E}[|C_{12}^{(n)}|^{2(2+\delta)/\delta}] = o(1)$ as $n \to \infty$. On the other hand, since the $\hat{R}_i^{(n)}(\boldsymbol{\theta}_0)$'s are the ranks of an exchangeable vector [see Assumption (A4)], we obtain that

$$\begin{aligned}
\frac{n(n-1)}{(n+1)^2}\mathrm{E}\big[|B_1^{(n)}B_2^{(n)}|^{2+\delta}\big] &= \frac{1}{(n+1)^2}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\bigg|J_{k,f_\star}\bigg(\frac{i}{n+1}\bigg)J_{k,f_\star}\bigg(\frac{j}{n+1}\bigg)\bigg|^{2+\delta} \\
&\le \Bigg(\frac{1}{n+1}\sum_{i=1}^{n+1}\bigg|J_{k,f_\star}\bigg(\frac{i}{n+1}\bigg)\bigg|^{2+\delta}\Bigg)^2;
\end{aligned}$$

this last sum is a Riemann sum for $\int_0^1 |J_{k,f_\star}(u)|^{2+\delta}\,du$, which is finite [Assumption (A3)]. Consequently, $\mathrm{E}[|B_1^{(n)}B_2^{(n)}|^{2+\delta}] = O(1)$ as $n \to \infty$, and the result follows. $\square$

PROOF OF PROPOSITION 3. From Lemma 3, we have, under $\mathcal{H}^{(n)}(\boldsymbol{\theta}_0, \mathbf{I}_k, f)$,

$$Q^{(n)}_{f_\star}(\boldsymbol{\theta}_0) = \big(\mathbf{T}^{(n)}_{f_\star;f}(\boldsymbol{\theta}_0)\big)'(\boldsymbol{\Gamma}_{\mathbf{I}_k,f_\star})^{-1}\mathbf{T}^{(n)}_{f_\star;f}(\boldsymbol{\theta}_0) + o^{(n)}_{\mathrm{P}}(1)$$
$$= \tilde{Q}^{(n)}_{f_\star;f}(\boldsymbol{\theta}_0) + o^{(n)}_{\mathrm{P}}(1),$$

with $\mathbf{T}^{(n)}_{f_\star;f}(\boldsymbol{\theta}_0)$ given in (11). The proof of the first part of Proposition 3 follows, since $\mathbf{T}^{(n)}_{f_\star;f}(\boldsymbol{\theta}_0)$ under $\mathcal{H}^{(n)}(\boldsymbol{\theta}_0, \mathbf{I}_k, f)$ is asymptotically $\mathcal{N}_k(\mathbf{0}, \boldsymbol{\Gamma}_{\mathbf{I}_k,f_\star})$, and since $Q^{(n)}_{f_\star}(\boldsymbol{\theta}_0)$ is affine-invariant.

Still from Lemma 3, $Q^{(n)}_{f_\star}(\boldsymbol{\theta}_0)$ is also asymptotically equivalent, under $\mathcal{H}^{(n)}(\boldsymbol{\theta}_0, \boldsymbol{\Sigma}, f)$, to $\tilde{Q}^{(n)}_{f_\star;(\boldsymbol{\Sigma},f)}(\boldsymbol{\theta}_0) := (\mathbf{T}^{(n)}_{f_\star;(\boldsymbol{\Sigma},f)}(\boldsymbol{\theta}_0))'\boldsymbol{\Gamma}^{-1}_{\boldsymbol{\Sigma},f_\star}\mathbf{T}^{(n)}_{f_\star;(\boldsymbol{\Sigma},f)}(\boldsymbol{\theta}_0)$, where

$$\mathbf{T}^{(n)}_{f_\star;(\boldsymbol{\Sigma},f)}(\boldsymbol{\theta}_0) := n^{-1/2}\sum_{i=1}^{n} J_{k,f_\star}\big(\tilde{F}_k\big(d^{(n)}_i(\boldsymbol{\theta}_0, \boldsymbol{\Sigma})\big)\big)\boldsymbol{\Sigma}^{-1/2}\mathbf{U}^{(n)}_i(\boldsymbol{\theta}, \boldsymbol{\Sigma}).$$

On the other hand, it is easy to see that, still under $\mathcal{H}^{(n)}(\boldsymbol{\theta}_0, \boldsymbol{\Sigma}, f)$, $\mathbf{T}^{(n)}_{f_\star;(\boldsymbol{\Sigma},f)}(\boldsymbol{\theta}_0)$ and the local log-likelihood $L^{(n)}_{\boldsymbol{\theta}_0+n^{-1/2}\boldsymbol{\tau}/\boldsymbol{\theta}_0;\boldsymbol{\Sigma},f}$ are jointly multinormal, with asymptotic covariance $\frac{1}{k}C_k(f_\star, f)\boldsymbol{\Sigma}^{-1}\boldsymbol{\tau}$; Le Cam's third Lemma thus implies that $\mathbf{T}^{(n)}_{f_\star;(\boldsymbol{\Sigma},f)}(\boldsymbol{\theta}_0)$ under $\mathcal{H}^{(n)}(\boldsymbol{\theta}_0 + n^{-1/2}\boldsymbol{\tau}, \boldsymbol{\Sigma}, f)$ is asymptotically $\mathcal{N}_k(\frac{1}{k}C_k(f_\star, f) \times \boldsymbol{\Sigma}^{-1}\boldsymbol{\tau}, \boldsymbol{\Gamma}_{\boldsymbol{\Sigma},f_\star})$. This establishes the second part of Proposition 3. □

PROOF OF PROPOSITION 5. Since the tests $\phi^{(n)}_{f_\star}$ and T^2 are both affine-invariant, we may assume that the underlying distribution is spherical. Under $\mathcal{H}^{(n)}(\boldsymbol{\theta}_0, \mathbf{I}_k, f)$, the Hotelling test statistic is asymptotically equivalent to $(\mathbf{T}^{(n)}(\boldsymbol{\theta}_0))'\boldsymbol{\Gamma}^{-1}_{\mathbf{I}_k}\mathbf{T}^{(n)}(\boldsymbol{\theta}_0)$, where

$$\mathbf{T}^{(n)}(\boldsymbol{\theta}_0) := n^{-1/2}\sum_{i=1}^{n} d^{(n)}_i(\boldsymbol{\theta}_0, \mathbf{I}_k)\mathbf{U}^{(n)}_i(\boldsymbol{\theta}_0, \mathbf{I}_k) = n^{-1/2}\sum_{i=1}^{n}(\mathbf{X}^{(n)}_i - \boldsymbol{\theta}_0)$$

and $\boldsymbol{\Gamma}_{\mathbf{I}_k} := \frac{1}{k}D_k(f)\mathbf{I}_k$, so that the same reasoning as in the proof of the second part of Proposition 3 implies that the Hotelling test statistic, under the local alternatives considered there, is asymptotically noncentral chi-square, with k degrees of freedom and noncentrality parameter

$$\frac{1}{k}\frac{E^2_k(f)}{D_k(f)}\boldsymbol{\tau}'\boldsymbol{\tau} \qquad \text{with } E_k(f) := \mathrm{E}\big[\tilde{F}^{-1}_k(U)J_{k,f}(U)\big].$$

This completes the proof, since the desired ARE values are obtained as the ratios of the corresponding noncentrality parameters. □

APPENDIX C

The Chernoff–Savage and Hodges–Lehmann properties.

PROOF OF PROPOSITION 6. The asymptotic relative efficiency of the van der Waerden test, with respect to Hotelling, under radial density f, is

$$\text{ARE}_{k,f}(\phi^{(n)}_{vdW}/T^2) = \frac{1}{k^3} D_k(f) \mathrm{E}^2\big[\tilde{\Phi}_k^{-1}(U) J_{k,f}(U)\big], \tag{14}$$

where, letting $\phi(r) := \exp(-r^2/2)$, $\tilde{\Phi}_k$ stands for the distribution function associated with $\tilde{\phi}_k(r) := (\mu_{k-1;\phi})^{-1} r^{k-1} \phi(r)\, I_{[r>0]}$. Without loss of generality, we restrict ourselves to the radial densities f satisfying $D_k(f) = \mathrm{E}[(\tilde{F}_k^{-1}(U))^2] = k$. Indeed, writing $f_a(r) := f(ar)$, $a > 0$, we have $\mu_{k;f_a} = a^{-(k+1)} \mu_{k;f}$, $D_k(f_a) = \mu_{k+1;f_a}/\mu_{k-1;f_a} = a^{-2} D_k(f)$ and $\text{ARE}_{k,f_a}(\phi^{(n)}_{vdW}/T^2) = \text{ARE}_{k,f}(\phi^{(n)}_{vdW}/T^2)$.

Thus, we only have to show that, for any $k \in \mathbb{N}_0$ and any f such that $D_k(f) = k$,

$$H_k(f) := \mathrm{E}\big[\tilde{\Phi}_k^{-1}(U) J_{k,f}(U)\big] \geq k,$$

with equality at $f = \phi$ only. This variational problem takes a simpler form after the following change of notation. First rewrite the functional H as

$$\begin{aligned}
H_k(f) &= \int_0^\infty \tilde{\Phi}_k^{-1}\big(\tilde{F}_k(r)\big)\varphi_f(r)\, \tilde{f}_k(r)\, dr \\
&= \frac{1}{\mu_{k-1;f}} \int_0^\infty \tilde{\Phi}_k^{-1}\big(\tilde{F}_k(r)\big)\big(-f'(r)\big) r^{k-1}\, dr \\
&= \int_0^\infty \bigg[\frac{1}{\tilde{\phi}_k(\tilde{\Phi}_k^{-1}(\tilde{F}_k(r)))}\, \tilde{f}_k(r) + \frac{k-1}{r} \tilde{\Phi}_k^{-1}\big(\tilde{F}_k(r)\big)\bigg] \tilde{f}_k(r)\, dr.
\end{aligned}$$

For any strictly positive (over $\mathbb{R}_0^+$) density f, the function $R : z \mapsto \tilde{F}_k^{-1} \circ \tilde{\Phi}_k(z)$ and its inverse $R^{-1} : r \mapsto \tilde{\Phi}_k^{-1} \circ \tilde{F}_k(r)$ are continuous monotone increasing transformations, mapping $\mathbb{R}_0^+$ onto itself, and satisfying $\lim_{z \downarrow 0} R(z) = \lim_{r \downarrow 0} R^{-1}(r) = 0$ and $\lim_{z \to \infty} R(z) = \lim_{r \to \infty} R^{-1}(r) = \infty$. Similarly, any continuous monotone increasing transformation R of $\mathbb{R}_0^+$ such that

$$\lim_{z \downarrow 0} R(z) = 0 \quad \text{and} \quad \lim_{z \to \infty} R(z) = \infty \tag{15}$$

characterizes a nonvanishing density f over $\mathbb{R}_0^+$ via the relation $R = \tilde{F}_k^{-1} \circ \tilde{\Phi}_k$. The variational problem just described thus consists of minimizing

$$\begin{aligned}
H_k(R) &= \int_0^\infty \bigg[\frac{1}{\tilde{\phi}_k(z)} \frac{\tilde{\phi}_k(z)}{R'(z)} + \frac{k-1}{R(z)} z\bigg] \tilde{\phi}_k(z)\, dz \\
&= \int_0^\infty \bigg[\frac{1}{R'(z)} + \frac{k-1}{R(z)} z\bigg] \tilde{\phi}_k(z)\, dz,
\end{aligned} \tag{16}$$

with respect to $R:\mathbb{R}_0^+ \to \mathbb{R}_0^+$ continuous and monotone increasing, under the constraints (15), since $\tilde{f}_k(r) = \frac{d}{dr}\tilde{F}_k(r) = \tilde{\phi}_k(z)/(\frac{d}{dz}R)$, and $\tilde{f}_k(r)\,dr = d\tilde{F}_k(r) = \tilde{\phi}_k(z)\,dz$. The constraint $D_k(f) = k$ now takes the form

$$D_k(R) = \int_0^\infty R^2(z)\tilde{\phi}_k(z)\,dz = k.$$

While complicating the form of $H_k(R)$ for $k > 1$, the second term in (16) does not affect its convexity, and this is the reason why the classical Chernoff–Savage argument extends to the multivariate context. Note that if $R_a := aR$ for $a > 0$, then $H_k(R_a) = a^{-1}H_k(R)$ and $D_k(R_a) = a^2 D_k(R)$. Using the same argument as in Lemma 1 of Chernoff and Savage (1958), this and the convexity of $H_k(R)$ imply that the solution R_1 of the minimization problem, if it exists, is unique. As in Chernoff and Savage (1958), the following lemma is a consequence of the convexity of $H_k(R)$.

LEMMA 5. *Let R_1 belong to the class $\mathcal{R}$ of monotone increasing and continuous functions $R:\mathbb{R}_0^+ \to \mathbb{R}_0^+$ such that* (15) *holds and $D_k(R) = k$. Then R_1 is a solution of the minimization problem under consideration if and only if, for any $R_2 \in \mathcal{R}$, there exists a $\xi \geq 0$ such that $H'_k(0) + \xi D'_k(0) \geq 0$, where*

$$H'_k(0) := \frac{d}{dw}\big(H_k\big((1-w)R_1 + wR_2\big)\big)\Big|_{w=0}$$

and

$$D'_k(0) := \frac{d}{dw}\big(D_k\big((1-w)R_1 + wR_2\big)\big)\Big|_{w=0}.$$

Now, it is easy to check that

$$\begin{aligned} H'_k(0) &= \int_0^\infty \biggl[-\frac{R'_2(z) - R'_1(z)}{(R'_1(z))^2} - \frac{(k-1)z(R_2(z) - R_1(z))}{(R_1(z))^2}\biggr]\tilde{\phi}_k(z)\,dz \\ &= \int_0^\infty \big(R_2(z) - R_1(z)\big)\biggl[\frac{\tilde{\phi}'_k(z)}{(R'_1(z))^2} - \frac{2\tilde{\phi}_k(z)R''_1(z)}{(R'_1(z))^3} - \frac{(k-1)z\tilde{\phi}_k(z)}{(R_1(z))^2}\biggr]dz \end{aligned}$$

and that

$$D'_k(0) = 2\int_0^\infty R_1(z)\big(R_2(z) - R_1(z)\big)\tilde{\phi}_k(z)\,dz,$$

so that, if $R_1(z) := z$ for all $z > 0$,

$$\begin{aligned} H'_k(0) + \xi D'_k(0) &= \int_0^\infty \big(R_2(z) - z\big)\biggl[\tilde{\phi}'_k(z) - \frac{(k-1)\tilde{\phi}_k(z)}{z} + 2\xi z\tilde{\phi}_k(z)\biggr]dz \\ &= \frac{1}{\mu_{k-1;\phi}}\int_0^\infty \big(R_2(z) - z\big)\big[z^{k-1}\phi'(z) + 2\xi z^k\phi(z)\big]\,dz \end{aligned}$$

which equals zero for $\xi = 1/2$. Lemma 5 thus implies that $f = \phi$ is the unique solution of the variational problem considered. □

We now turn to the proof of the multivariate extension of the Hodges–Lehmann theorem.

PROOF OF PROPOSITION 7. First note that, from Proposition 5,

$$\mathrm{ARE}_{k,f}(\phi_W^{(n)}/T^2) = \frac{3}{k^2} D_k(f)\,\mathrm{E}^2[U J_{k,f}(U)].$$

As in the proof of Proposition 6, it is clear (by considering $f_a(r) := f(ar)$, $a > 0$) that we may assume that $D_k(f) = 1$. Therefore, the problem reduces to the variational problem

$$\inf_{f \in \mathcal{C}} \mathrm{E}[U J_{k,f}(U)], \qquad \text{with } \mathcal{C} := \{f \mid D_k(f) = 1\}. \tag{17}$$

Integrating by parts, we obtain

$$\begin{aligned}\mathrm{E}[U J_{k,f}(U)] &= \int_0^\infty \tilde{F}_k(r)\varphi_f(r)\,\tilde{f}_k(r)\,dr\\ &= \int_0^\infty \left[\big(\tilde{f}_k(r)\big)^2 + \frac{k-1}{r}\tilde{F}_k(r)\,\tilde{f}_k(r)\right]dr,\end{aligned}$$

so that (17) in turn is equivalent to

$$\inf_{\tilde{f} \in \tilde{\mathcal{C}}} \int_0^\infty \left[\big(\tilde{f}_k(r)\big)^2 + \frac{k-1}{r}\tilde{F}_k(r)\,\tilde{f}_k(r)\right]dr, \tag{18}$$

where $\tilde{\mathcal{C}}$ is the set of all $\tilde{f}$ defined on $\mathbb{R}_0^+$ such that

$$\int_0^\infty \tilde{f}_k(r)\,dr = \int_0^\infty r^2\,\tilde{f}_k(r)\,dr = 1. \tag{19}$$

Substituting $y, \dot{y}$ and t for $\tilde{F}_k$, $\tilde{f}_k$ and r, respectively, the Euler–Lagrange equation associated with the variational problem (18), (19) takes the form

$$t^2\ddot{y} - \frac{k-1}{2}y = -\lambda_2 t^3, \tag{20}$$

where λ_2 stands for the Lagrange multiplier associated with the second constraint in (19). Letting $y = t^{1/2}u$, equation (20) reduces to the Euler equation

$$t^2\ddot{u} + t\dot{u} - \frac{2k-1}{4}u = -\lambda_2 t^{5/2}.$$

Finally, we obtain that the general solution of (20) is given by

$$y(t) = \frac{2\lambda_2}{k-13}t^3 + \alpha t^{(1+\sqrt{2k-1})/2} + \beta t^{(1-\sqrt{2k-1})/2}$$

for $k \neq 13$, and

$$y(t) = \frac{\lambda_2}{25}(1 - 5 \ln t)t^3 + \alpha t^3 + \beta t^{-2}$$

for $k = 13$. Since $y(0+) = 0$, it is clear that $\beta = 0$, irrespective of the dimension k. On the other hand, $\dot{y}(t) \geq 0$ for small t implies that $\alpha \geq 0$ for $k < 13$, and $\lambda_2 \geq 0$ for $k \geq 13$. So that $\int \dot{y} = 1$ yields $\lambda_2 > 0$ for $k < 13$ and $\alpha < 0$ for $k > 13$. Therefore $\dot{y} \geq 0$ implies that $\dot{y}$ is compactly supported in $\mathbb{R}_0^+$, with support $[0, a]$, say.

It follows from the constraints (19) and the continuity of $\dot{y}$ that the extremals of the variational problem under study are the solutions of (20) that satisfy

$$y(a) = 1, \quad \dot{y}(a) = 0 \quad \text{and} \quad \int_0^a t^2 \dot{y}(t)\, dt = 1. \tag{21}$$

It is then a simple exercise to check that conditions (21) yield, for $k \neq 13$,

$$\lambda_2 = \frac{3\sqrt{3}}{10\sqrt{5}}(k-13)\frac{(\sqrt{2k-1}+1)^{5/2}}{(\sqrt{2k-1}-5)(\sqrt{2k-1}+5)^{3/2}},$$

$$a = \left(\frac{5}{3}\frac{\sqrt{2k-1}+5}{\sqrt{2k-1}+1}\right)^{1/2},$$

$$\alpha = \frac{-6}{\sqrt{2k-1}-5}\left(\frac{3}{5}\frac{\sqrt{2k-1}+1}{\sqrt{2k-1}+5}\right)^{(\sqrt{2k-1}+1)/4},$$

and for $k = 13$,

$$\lambda_2 = 81/25, \quad a = 5/3, \qquad \alpha = \frac{405 \ln(5/3) + 54}{625}.$$

To conclude, note that, integrating by parts and using (20),

$$\begin{aligned}
&\min_f \mathrm{E}[U J_{k,f}(U)] \\
&\qquad = \int_0^\infty \left[(\dot{y}(t))^2 + \frac{k-1}{t} y(t)\dot{y}(t)\right] dt \\
&\qquad = \int_0^\infty \left[-2t\dot{y}(t)\ddot{y}(t) + \frac{k-1}{t} y(t)\dot{y}(t)\right] dt \\
&\qquad = \int_0^\infty 2\lambda_2 t^2 \dot{y}(t)\, dt = 2\lambda_2,
\end{aligned}$$

so that $\min_f \mathrm{ARE}_{k,f}(\phi_W^{(n)}/T^2) = \frac{12}{k^2}(\lambda_2)^2$. This completes the proof of Proposition 7. $\square$

REMARK. As an immediate corollary, we also obtain that the infimum in Proposition 7 is reached (for fixed k) at the collection of radial densities f for which $\tilde{f}_k$ is in $\{\tilde{f}_{k,\sigma}(r) := \sigma^{-1}\tilde{f}_{k,1}(\sigma^{-1}r)\}$, where

$$\tilde{f}_{k,1}(r) = \Bigg(\frac{9\sqrt{3}}{5\sqrt{5}} \frac{(\sqrt{2k-1}+1)^{5/2}}{(\sqrt{2k-1}-5)(\sqrt{2k-1}+5)^{3/2}} r^2 \\ - 3\frac{\sqrt{2k-1}+1}{\sqrt{2k-1}-5}\left(\frac{3}{5}\frac{\sqrt{2k-1}+1}{\sqrt{2k-1}+5}\right)^{(\sqrt{2k-1}+1)/4} r^{(\sqrt{2k-1}-1)/2}\Bigg) \\ \times I\left[0 < r < \left(\frac{5}{3}\frac{\sqrt{2k-1}+5}{\sqrt{2k-1}+1}\right)^{1/2}\right],$$

for $k \neq 13$, and

$$\tilde{f}_{13,1}(r) = \frac{243}{125}\left(\ln\frac{5}{3} - \ln r\right) r^2 I\left[0 < r < \frac{5}{3}\right].$$

See Figure 4 for the graphs of the densities $\tilde{f}_{k,1}$ for some values of the space dimension k.

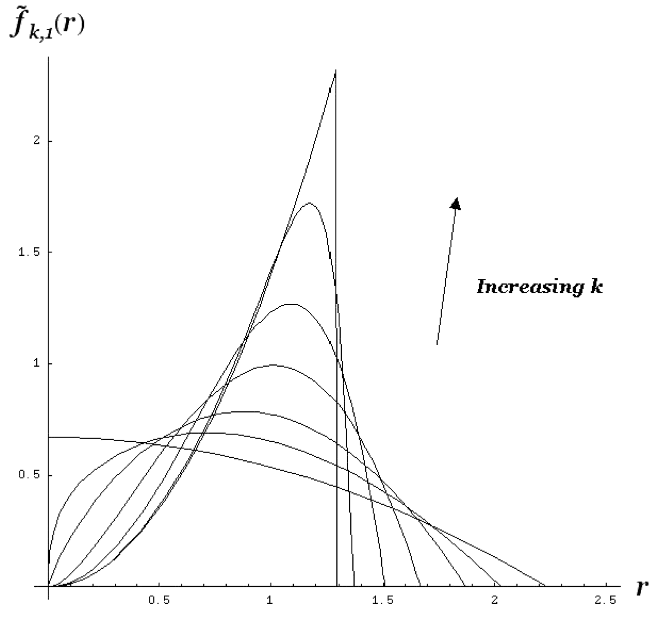

FIG. 4. *Graphs of the densities $\tilde{f}_{k,1}$ at which the infimum of the AREs of Wilcoxon-type tests with respect to Hotelling's is reached, for dimensions $k = 1, 2, 4, 13, 50, 500$ and 10^{10}, respectively, of the observations.*

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