Multivariate Signed Ranks : Randles' Interdirections or Tyler's Angles?

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Abstract. Hallin and Paindaveine (2002a) developed, for the multivariate (elliptically symmetric) one-sample location problem, a class of optimal procedures, based on Randles' interdirections and the ranks of pseudo-Mahalanobis distances. We present an alternative version of these procedures in which interdirections are replaced by "Tyler angles", namely, the angles between the observations standardized via Tyler's estimator of scatter. These Tyler angles are indeed computationally preferable (in terms of CPU time) to interdirections. We show that the two approaches are asymptotically equivalent. A Monte-Carlo study is conducted to compare their small-sample efficiency and robustness features. Simulations indicate that, whereas interdirections and Tyler angles yield comparable results under strict ellipticity and *radial* outliers, interdirections are significantly more reliable in the presence of *angular* outliers. This study is focused on the simple one-sample location problem. It readily extends, with obvious changes, to more complex models such as multivariate regression or analysis of variance, and to time series models (see Randles and Um 1998, Hallin and Paindaveine 2002b).

1. Introduction.

Denote by $(\mathbf{X}_1^{(n)}, \ldots, \mathbf{X}_n^{(n)})$ an observed $n-$tuple of i.i.d. k-variate elliptically symmetric variables, with center of symmetry $\boldsymbol{\theta}$. More precisely, we throughout assume that the probability density function at $(\mathbf{X}_1^{(n)}, \ldots, \mathbf{X}_n^{(n)}) \in \mathbb{R}^{nk}$ of the sample is of the form

$$\prod_{i=1}^{n} \underline{f}\left(\mathbf{X}_i^{(n)}; \boldsymbol{\theta}, \boldsymbol{\Sigma}, f\right) := \frac{(c_{k,f})^n}{(\det \boldsymbol{\Sigma})^{n/2}} \prod_{i=1}^{n} f\left(d_i^{(n)}(\boldsymbol{\theta}, \boldsymbol{\Sigma})\right), \tag{1.1}$$

where $c_{k,f} > 0$ is a normalization factor, the *shape matrix* $\boldsymbol{\Sigma}$ is some $k \times k$ symmetric and positive-definite matrix, the *radial density* f is a function that maps

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$\mathbb{R}_0^+$ into itself, satisfying $\int_0^\infty r^{k-1} f(r)\, dr < \infty$, and

$$d_i^{(n)}(\boldsymbol{\theta}, \boldsymbol{\Sigma}) := \Big((\mathbf{X}_i^{(n)} - \boldsymbol{\theta})' \boldsymbol{\Sigma}^{-1} (\mathbf{X}_i^{(n)} - \boldsymbol{\theta}) \Big)^{1/2}$$

denotes the distance between $\mathbf{X}_i^{(n)}$ and $\boldsymbol{\theta}$ in the metric associated with $\boldsymbol{\Sigma}$. We denote by $\mathcal{H}^{(n)}(\boldsymbol{\theta}, \boldsymbol{\Sigma}, f)$ the hypothesis under which the sample has p.d.f. (1.1). "Parameters" $\boldsymbol{\Sigma}$ and f are only identified up to a scalar factor, but this will have no influence on the procedures we will consider.

With this notation, we want to test the hypothesis $\mathcal{H}_0^{(n)}$ under which $\boldsymbol{\theta}$ is equal to some fixed value $\boldsymbol{\theta}_0 \in \mathbb{R}^k$, without specifying any further the underlying elliptically symmetric distribution (i.e., without specifying $\boldsymbol{\Sigma}$ and f).

The classical procedure for this problem is the Hotelling test. Letting

$$\bar{\mathbf{X}}^{(n)} := \frac{1}{n} \sum_{i=1}^n \mathbf{X}_i^{(n)} \quad \text{and} \quad \mathbf{S}^{(n)} := \frac{1}{n-1} \sum_{i=1}^n (\mathbf{X}_i^{(n)} - \bar{\mathbf{X}}^{(n)})(\mathbf{X}_i^{(n)} - \bar{\mathbf{X}}^{(n)})',$$

this test, which requires the underlying distribution to be Gaussian, rejects the null hypothesis whenever

$$\frac{n-k}{k(n-1)} T^{2(n)} := \frac{n(n-k)}{k(n-1)} (\bar{\mathbf{X}}^{(n)} - \boldsymbol{\theta}_0)' (\mathbf{S}^{(n)})^{-1} (\bar{\mathbf{X}}^{(n)} - \boldsymbol{\theta}_0),$$

exceeds the α-upper quantile $F_{k,n-k;1-\alpha}$ of a Fisher distribution with k and $(n-k)$ degrees of freedom. Under asymptotic form, the same test rejects the null hypothesis whenever $T^{2(n)} > \chi^2_{k;1-\alpha}$, where $\chi^2_{k;1-\alpha}$ denotes the α-upper quantile of a chi-square distribution with k degrees of freedom; this latter version is asymptotically valid under any finite-variance elliptically symmetric distribution. This test is optimal (in its asymptotic version, asymptotically optimal) under Gaussian densities.

Hallin and Paindaveine (2002a) proposed a class of multivariate signed-rank competitors to Hotelling's procedure, that are based on Randles' interdirections and on the ranks of pseudo-Mahalanobis distances between the observations and $\boldsymbol{\theta}_0$ (see Section 2). These multivariate signed-rank tests are valid under a very wide class of elliptically symmetric densities (in particular, they do not require any finite moments), and are locally and asymptotically optimal (LAO) if the score function is correctly specified. They often yield excellent overall performances : for instance, in their van der Waerden version, they uniformly dominate (in the Pitman sense) Hotelling's test (see Proposition 6 in Hallin and Paindaveine 2002a).

In this paper, we propose a closely related class of testing procedures in which Randles' interdirections are replaced by the angles between standardized residuals based on Tyler's estimator of scatter. The resulting tests (call them *angle-based tests*) are valid under the same class of densities as the *interdirection-based* ones. These two classes of tests are shown to be asymptotically equivalent. In particular, they share the same asymptotic efficiencies. A Monte-Carlo study is performed in order to compare their finite-sample behaviours and robustness features. The plain *sign-test* version of our interdirection-based procedure was considered by Randles (1989), its angle-based counterpart in Randles (2000).

This study is focused on the simple one-sample location problem. It readily extends, with obvious changes, to more complex models such as multivariate regression or analysis of variance, and to time series models (see Randles and Um 1998, Hallin and Paindaveine 2002b).

2. Interdirection-based procedures.

2.1. Local asymptotic optimality.

Local asymptotic normality (for fixed $\mathbf{\Sigma}$ and f) of the location model described in Section 1 is ensured by a few mild regularity conditions on the radial density f. More precisely, throughout the paper we assume that

(A) $f^{1/2}$ has a weak derivative $(f^{1/2})'$ (in the sense of distributions) in $L^2(\mathbb{R}_0^+)$, that satisfies $\int_0^\infty r^{k-1}((f^{1/2})'(r))^2\, dr < \infty$.

Let $\varphi_f := -2(f^{1/2})'/f^{1/2}$. Under Assumption (A), the radial Fisher information $\mathcal{I}_{k,f} := \mathrm{E}[\varphi_f^2(d_1^{(n)}(\boldsymbol{\theta}_0, \mathbf{\Sigma}))]$ is finite (expectation is taken under $\mathcal{H}^{(n)}(\boldsymbol{\theta}_0, \mathbf{\Sigma}, f)$).

With $d_i^{(n)}(\boldsymbol{\theta}_0, \mathbf{\Sigma})$ defined in (1.1), let $\mathbf{U}_i^{(n)}(\boldsymbol{\theta}_0, \mathbf{\Sigma}) := \mathbf{\Sigma}^{-1/2}(\mathbf{X}_i^{(n)} - \boldsymbol{\theta}_0)/d_i^{(n)}(\boldsymbol{\theta}_0, \mathbf{\Sigma})$ denote the *normalized standardized residual* associated with observation $\mathbf{X}_i^{(n)}$. Hallin and Paindaveine (2002a) show that a LAO procedure for testing $\boldsymbol{\theta} = \boldsymbol{\theta}_0$ under specified $\mathbf{\Sigma}$ and f is obtained by rejecting the null hypothesis whenever the quadratic statistic

$$Q_{\mathbf{\Sigma},f}^{(n)} := \frac{k}{n\,\mathcal{I}_{k,f}} \sum_{i,j=1}^{n} \varphi_f(d_i^{(n)}(\boldsymbol{\theta}_0, \mathbf{\Sigma}))\, \varphi_f(d_j^{(n)}(\boldsymbol{\theta}_0, \mathbf{\Sigma}))\ (\mathbf{U}_i^{(n)}(\boldsymbol{\theta}_0, \mathbf{\Sigma}))'\mathbf{U}_j^{(n)}(\boldsymbol{\theta}_0, \mathbf{\Sigma})$$

exceeds the α-upper quantile $\chi^2_{k,1-\alpha}$ of a chi-square distribution with k degrees of freedom; more precisely, this test is locally asymptotically maximin, at asymptotic level α, for $\mathcal{H}^{(n)}(\boldsymbol{\theta}_0, \mathbf{\Sigma}, f)$ against alternatives of the form $\bigcup_{\boldsymbol{\theta} \neq \boldsymbol{\theta}_0} \mathcal{H}^{(n)}(\boldsymbol{\theta}, \mathbf{\Sigma}, f)$.

Consequently, parametric LAO procedures can be based on the distances $d_i^{(n)}(\boldsymbol{\theta}_0, \mathbf{\Sigma})$ and the scalar products $(\mathbf{U}_i^{(n)}(\boldsymbol{\theta}_0, \mathbf{\Sigma}))'\mathbf{U}_j^{(n)}(\boldsymbol{\theta}_0, \mathbf{\Sigma})$, i.e., the cosines of the angles between standardized residuals. Hallin and Paindaveine (2002a) then propose to build nonparametric LAO procedures by replacing these angles by a non-parametric estimator based on Randles' interdirections, and the $d_i^{(n)}(\boldsymbol{\theta}_0, \mathbf{\Sigma})$'s by a function of their ranks. We now briefly describe these statistics.

2.2. Randles' interdirections.

Denote by $\mathcal{P}_{r_1,\ldots,r_{k-1}}(\boldsymbol{\theta}_0)$ the hyperplane going through $\mathbf{X}_{r_1}^{(n)} - \boldsymbol{\theta}_0, \ldots, \mathbf{X}_{r_{k-1}}^{(n)} - \boldsymbol{\theta}_0$ and the origin in $\mathbb{R}^k$. Let $D_{ij}^{(n)}(\mathcal{P}_{r_1,\ldots,r_{k-1}}(\boldsymbol{\theta}_0))$ be

- "1" if $\mathbf{X}_i^{(n)} - \boldsymbol{\theta}_0$ and $\mathbf{X}_j^{(n)} - \boldsymbol{\theta}_0$ lie on opposite sides of $\mathcal{P}_{r_1,\ldots,r_{k-1}}(\boldsymbol{\theta}_0)$,
- "1/2" if at least one of $\mathbf{X}_i^{(n)} - \boldsymbol{\theta}_0$ and $\mathbf{X}_j^{(n)} - \boldsymbol{\theta}_0$ is on $\mathcal{P}_{r_1,\ldots,r_{k-1}}(\boldsymbol{\theta}_0)$, and

- "0" if both $\mathbf{X}_i^{(n)} - \boldsymbol{\theta}_0$ and $\mathbf{X}_j^{(n)} - \boldsymbol{\theta}_0$ lie on the same side of $\mathcal{P}_{r_1,\ldots,r_{k-1}}(\boldsymbol{\theta}_0)$.

The interdirection $q_{ij}^{(n)}(\boldsymbol{\theta}_0)$ associated with the pair $(\mathbf{X}_i^{(n)} - \boldsymbol{\theta}_0, \mathbf{X}_j^{(n)} - \boldsymbol{\theta}_0)$ in the residual sample $(\mathbf{X}_1^{(n)} - \boldsymbol{\theta}_0, \ldots, \mathbf{X}_n^{(n)} - \boldsymbol{\theta}_0)$ is then defined as $\sum D_{ij}^{(n)}(\mathcal{P}_{r_1,\ldots,r_{k-1}}(\boldsymbol{\theta}_0))$ (where the sum runs over the $\binom{n}{k-1}$ possible hyperplanes $\mathcal{P}_{r_1,\ldots,r_{k-1}}(\boldsymbol{\theta}_0)$) for $i \neq j$, as 0 for $i = j$. Letting $p_{ij}^{(n)} := p_{ij}^{(n)}(\boldsymbol{\theta}_0) := q_{ij}^{(n)}(\boldsymbol{\theta}_0)/\binom{n}{k-1}$, the quantity $0 \leq p_{ij}^{(n)} \leq 1$ is essentially the proportion of such (data-based) hyperplanes that separate $\mathbf{X}_i^{(n)} - \boldsymbol{\theta}_0$ and $\mathbf{X}_j^{(n)} - \boldsymbol{\theta}_0$ (note that "counting 1/2" in the second bullet above is only a small-sample correction for the Randles (1989) sign test to admit Blumen (1958)'s bivariate sign test as a special case). Interdirections are invariant under linear transformations and under (any) radial transformations, so that they indifferently can be computed from the $\mathbf{X}_i^{(n)} - \boldsymbol{\theta}_0$'s as from the $\mathbf{U}_i^{(n)}(\boldsymbol{\theta}_0, \boldsymbol{\Sigma})$'s. In view of the consistency result in Lemma 2.1 below, (normalized) interdirections provide affine-invariant estimations of the Euclidean angles between the $\mathbf{U}_i^{(n)}(\boldsymbol{\theta}, \boldsymbol{\Sigma})$'s, that is, they estimate the quantities $\pi^{-1}\arccos((\mathbf{U}_i^{(n)}(\boldsymbol{\theta}, \boldsymbol{\Sigma}))'\mathbf{U}_j^{(n)}(\boldsymbol{\theta}, \boldsymbol{\Sigma}))$; see Hallin and Paindaveine (2002a) for a proof.

Lemma 2.1. *Let* $(\mathbf{U}_1, \mathbf{U}_2, \ldots)$ *be an i.i.d. process of random vectors that are uniformly distributed over the unit sphere in* $\mathbb{R}^k$. *For any* $\mathbf{v}$ *and* $\mathbf{w}$ *in* $\mathbb{R}^k$, *denote by* $\alpha(\mathbf{v}, \mathbf{w}) := \arccos(\mathbf{v}'\mathbf{w}/(\|\mathbf{v}\|\|\mathbf{w}\|))$ *the angle between* $\mathbf{v}$ *and* $\mathbf{w}$, *and by* $q^{(n)}(\mathbf{v}, \mathbf{w})$ *the interdirection associated with* $\mathbf{v}$ *and* $\mathbf{w}$ *in the sample* $(\mathbf{U}_1, \ldots, \mathbf{U}_n)$. *Then,* $p^{(n)}(\mathbf{v}, \mathbf{w}) := q^{(n)}(\mathbf{v}, \mathbf{w})/\binom{n}{k-1}$ *converges in quadratic mean to* $\pi^{-1}\alpha(\mathbf{v}, \mathbf{w})$ *as* $n \to \infty$.

2.3. Ranks of Mahalanobis distances.

As usual in rank-based inference, we intend to replace the distances $d_i^{(n)}(\boldsymbol{\theta}_0, \boldsymbol{\Sigma})$ by $\tilde{F}_k^{-1}(R_i^{(n)}(\boldsymbol{\theta}_0, \boldsymbol{\Sigma})/(n+1))$, where $R_i^{(n)}(\boldsymbol{\theta}_0, \boldsymbol{\Sigma})$ denotes the rank of $d_i^{(n)}(\boldsymbol{\theta}_0, \boldsymbol{\Sigma})$ among $d_1^{(n)}(\boldsymbol{\theta}_0, \boldsymbol{\Sigma}), \ldots, d_n^{(n)}(\boldsymbol{\theta}_0, \boldsymbol{\Sigma})$, and $\tilde{F}_k$ stands for the cumulative distribution function of $d_i^{(n)}(\boldsymbol{\theta}_0, \boldsymbol{\Sigma})$ under $\mathcal{H}^{(n)}(\boldsymbol{\theta}_0, \boldsymbol{\Sigma}, f)$.

Of course, the shape matrix $\boldsymbol{\Sigma}$ is generally unknown, and has to be estimated. Consequently, only the estimated ranks $\hat{R}_i^{(n)} := \hat{R}_i^{(n)}(\boldsymbol{\theta}_0)$ of $d_i^{(n)}(\boldsymbol{\theta}_0, \hat{\boldsymbol{\Sigma}})$ among $d_1^{(n)}(\boldsymbol{\theta}_0, \hat{\boldsymbol{\Sigma}}), \ldots, d_n^{(n)}(\boldsymbol{\theta}_0, \hat{\boldsymbol{\Sigma}})$ are available. As an estimate for $\boldsymbol{\Sigma}$, Hallin and Paindaveine (2002a) propose Tyler's estimator of scatter (see Tyler 1987). The resulting distances are called *pseudo*-Mahalanobis distances, in order to stress that they are not computed, as classical Mahalanobis distances, from the sample covariance matrix (which may not be root-n consistent). Since Tyler's estimator will play a crucial role in the procedures we propose in this paper, we briefly recall its definition and main properties.

Tyler's estimator of scatter is defined as $\widehat{\boldsymbol{\Sigma}}^{(n)} := (\mathbf{C}_{Tyl}^{(n)\prime}\mathbf{C}_{Tyl}^{(n)})^{-1}$, where $\mathbf{C}_{Tyl}^{(n)} := \mathbf{C}_{Tyl}^{(n)}(\mathbf{X}^{(n)} - \boldsymbol{\theta}_0)$ denotes the (unique for $n > k(k-1)$) upper triangular $k \times k$ matrix with positive diagonal elements and a "1" in the upper left corner that satisfies

$$\frac{1}{n}\sum_{i=1}^{n}\left(\frac{\mathbf{C}_{Tyl}^{(n)}(\mathbf{X}_i^{(n)}-\boldsymbol{\theta}_0)}{\|\mathbf{C}_{Tyl}^{(n)}(\mathbf{X}_i^{(n)}-\boldsymbol{\theta}_0)\|}\right)\left(\frac{\mathbf{C}_{Tyl}^{(n)}(\mathbf{X}_i^{(n)}-\boldsymbol{\theta}_0)}{\|\mathbf{C}_{Tyl}^{(n)}(\mathbf{X}_i^{(n)}-\boldsymbol{\theta}_0)\|}\right)' = \frac{1}{k}\,\mathbf{I}_k, \tag{2.1}$$

($\mathbf{I}_k$ stands for the k-dimensional identity matrix). Tyler's estimator is root-n consistent, up to a multiplicative factor, for the shape matrix $\boldsymbol{\Sigma}$. More precisely, there exists a positive real a such that $\sqrt{n}(\widehat{\boldsymbol{\Sigma}}^{(n)} - a\,\boldsymbol{\Sigma})$ is $O_{\mathrm{P}}(1)$ as $n\to\infty$ under $\mathcal{H}^{(n)}(\boldsymbol{\theta}_0,\boldsymbol{\Sigma},.) := \bigcup_f \mathcal{H}^{(n)}(\boldsymbol{\theta}_0,\boldsymbol{\Sigma},f)$. The matrix $\mathbf{C}_{Tyl}^{(n)}$ (hence also Tyler's estimator) is clearly invariant under permutations and reflections with respect to the origin in $\mathbb{R}^k$ of the residuals $\mathbf{X}_i^{(n)}-\boldsymbol{\theta}_0$, and has the following affine-equivariance properties : denoting by $\mathbf{C}_{Tyl}^{(n)}(\mathbf{M})$ the statistics $\mathbf{C}_{Tyl}^{(n)}$ computed from the sample $(\mathbf{M}(\mathbf{X}_1^{(n)}-\boldsymbol{\theta}_0),\ldots,\mathbf{M}(\mathbf{X}_n^{(n)}-\boldsymbol{\theta}_0))$, we have $\mathbf{C}_{Tyl}^{(n)}(\mathbf{M}) = d\,\mathbf{O}\mathbf{C}_{Tyl}^{(n)}\mathbf{M}^{-1}$ for some orthogonal matrix $\mathbf{O}$ and some scalar d that depends on the sample $(\mathbf{X}_i^{(n)}-\boldsymbol{\theta}_0$, $i=1,\ldots,n)$ but not on index i. The affine-invariance of pseudo-Mahalanobis ranks follows.

2.4. A class of interdirection-based procedures.

Let K be a continuous function in $L^{2+\delta}(]0,1[)$, for some $\delta>0$, that can be expressed as the difference of two monotone increasing functions. This assumption of monotonicity plays a role in the proof of the asymptotic representation result (2.3). A similar assumption is made by Hájek in his classical asymptotic representation results for rank statistics in the one-dimensional setup. The K-score version of the test described in Section 2.1 is based on the statistics

$$Q_K^{(n)} := \frac{k}{n\,\mathrm{E}[K^2(U)]}\sum_{i,j=1}^{n} K\Big(\frac{\hat{R}_i^{(n)}}{n+1}\Big)\,K\Big(\frac{\hat{R}_j^{(n)}}{n+1}\Big)\cos(\pi p_{ij}^{(n)}), \tag{2.2}$$

(U is uniform over $]0,1[$) to be compared with the same critical values as in Section 2.1. Optimality under fixed radial density $f_\star$ will be obtained with the score function $K_{f_\star} := \varphi_{f_\star}\circ\tilde{F}_{\star k}^{-1}$, where $f_\star:\mathbb{R}_0^+\to\mathbb{R}_0^+$ satisfies Assumption (A). We denote by $Q_{f_\star}^{(n)}$ (instead of $Q_{K_{f_\star}}^{(n)}$) the corresponding test statistics. The test $\phi_K^{(n)}$ has a number of highly desirable properties, as shown by the following proposition.

Proposition 2.2. *Consider the sequence of tests $\phi_K^{(n)}$ (resp. $\phi_{f_\star}^{(n)}$) rejecting the null hypothesis $\mathcal{H}_0^{(n)}$ whenever $Q_K^{(n)}$ (resp. $Q_{f_\star}^{(n)}$) exceeds the α-upper quantile $\chi^2_{k,1-\alpha}$ of a chi-square distribution with k degrees of freedom. Then,*

(i) *$Q_K^{(n)}$ is strictly affine-invariant and asymptotically invariant under the group of continuous monotone radial transformations, and*
(ii) *is asymptotically chi-square with k degrees of freedom under $\mathcal{H}^{(n)}(\boldsymbol{\theta}_0,.,.)$;*
(iii) *the sequence $\phi_K^{(n)}$ has asymptotic level α;*

(iv) *the sequence* $\phi^{(n)}_{f_\star}$ *is locally asymptotically maximin, at asymptotic level* α, *for* $\mathcal{H}^{(n)}(\boldsymbol{\theta}_0, ., .)$ *against alternatives of the form* $\bigcup_{\boldsymbol{\theta}\neq\boldsymbol{\theta}_0} \bigcup_{\boldsymbol{\Sigma}} \mathcal{H}^{(n)}(\boldsymbol{\theta}, \boldsymbol{\Sigma}, f_\star)$.

Part (i) of the proposition results from the invariance properties of interdirections and pseudo-Mahalanobis ranks. The other statements are easily obtained from the following asymptotic representation result : under $\mathcal{H}^{(n)}(\boldsymbol{\theta}_0, \boldsymbol{\Sigma}, f)$,

$$Q^{(n)}_K = \tilde{Q}^{(n)}_{K;\boldsymbol{\Sigma},f} + o_{\mathrm{P}}(1), \tag{2.3}$$

where

$$\tilde{Q}^{(n)}_{K;\boldsymbol{\Sigma},f} := \frac{k}{n\,E[K^2(U)]} \sum_{i,j=1}^{n} K\left(\tilde{F}_k(d^{(n)}_i(\boldsymbol{\theta}_0, \boldsymbol{\Sigma}))\right) K\left(\tilde{F}_k(d^{(n)}_j(\boldsymbol{\theta}_0, \boldsymbol{\Sigma}))\right) \times (\mathbf{U}^{(n)}_i(\boldsymbol{\theta}_0, \boldsymbol{\Sigma}))'\mathbf{U}^{(n)}_j(\boldsymbol{\theta}_0, \boldsymbol{\Sigma}).$$

See Hallin and Paindaveine (2002a) for a proof. Note that (2.3) implies that the interdirection-based test statistic $Q^{(n)}_{f_\star}$ is asymptotically equivalent, under $\mathcal{H}^{(n)}(\boldsymbol{\theta}_0, \boldsymbol{\Sigma}, f_\star)$ and contiguous alternatives, to the test statistic $Q^{(n)}_{\boldsymbol{\Sigma},f_\star}$ of the parametric LAO test (which yields part (iv) in the proposition).

3. Angle-based procedures.

3.1. Tyler angles

Since the shape matrix in (2.2) has to be estimated anyway, one could think of estimating the $\mathbf{U}^{(n)}_i(\boldsymbol{\theta}_0, \boldsymbol{\Sigma})$'s by means of the *Tyler residuals* $\mathbf{W}^{(n)}_i := \mathbf{W}^{(n)}_i(\boldsymbol{\theta}_0) := \mathbf{C}^{(n)}_{Tyl}(\mathbf{X}^{(n)}_i - \boldsymbol{\theta}_0)/\|\mathbf{C}^{(n)}_{Tyl}(\mathbf{X}^{(n)}_i - \boldsymbol{\theta}_0)\|$. Then, natural estimates of the scalar products $(\mathbf{U}^{(n)}_i(\boldsymbol{\theta}_0, \boldsymbol{\Sigma}))'\mathbf{U}^{(n)}_j(\boldsymbol{\theta}_0, \boldsymbol{\Sigma})$ are provided by the *Tyler cosines* $(\mathbf{W}^{(n)}_i)'\mathbf{W}^{(n)}_j$. The following consistency result is proved in Hallin and Paindaveine (2002c).

Lemma 3.1. *For all* $i = 1, \ldots, n$, $\mathbf{W}^{(n)}_i = \mathbf{O}\mathbf{U}^{(n)}_i(\boldsymbol{\theta}_0, \boldsymbol{\Sigma}) + O_{\mathrm{P}}(n^{-1/2})$ *as* $n \to \infty$, *under* $\bigcup_f \mathcal{H}^{(n)}(\boldsymbol{\theta}_0, \boldsymbol{\Sigma}, f)$, *for some orthogonal matrix* $\mathbf{O}$.

The orthogonal matrix $\mathbf{O}$ in the lemma of course does not depend on i, or the claim would be trivial. Note that the $\mathbf{W}^{(n)}_i$'s remain unchanged if the residuals $\mathbf{X}^{(n)}_1 - \boldsymbol{\theta}_0, \ldots, \mathbf{X}^{(n)}_n - \boldsymbol{\theta}_0$ are moved radially from the origin in $\mathbb{R}^k$, so that Tyler residuals are invariant under continuous monotone radial transformations. The affine-equivariance result below trivially follows from the affine-equivariance property of $\mathbf{C}^{(n)}_{Tyl}$.

Lemma 3.2. *Denote by* $\mathbf{W}^{(n)}_i(\mathbf{M})$ *the Tyler residual associated with observation* i *when computed from the transformed residuals* $\mathbf{M}(\mathbf{X}^{(n)}_1 - \boldsymbol{\theta}_0), \ldots, \mathbf{M}(\mathbf{X}^{(n)}_n - \boldsymbol{\theta}_0)$. *Then,* $\mathbf{W}^{(n)}_i(\mathbf{M}) = \mathbf{O}\mathbf{W}^{(n)}_i$, *where* $\mathbf{O}$ *is the orthogonal matrix that is involved in the equivariance relation* $\mathbf{C}^{(n)}_{Tyl}(\mathbf{M}) = d\,\mathbf{O}\,\mathbf{C}^{(n)}_{Tyl}\mathbf{M}^{-1}$.

Lemma 3.2 implies that any orthogonally-invariant function of the Tyler residuals is strictly affine-invariant. In particular, the cosines of Euclidean angles between the $\mathbf{W}_i^{(n)}$'s, i.e., the $((\mathbf{W}_i^{(n)})'\mathbf{W}_j^{(n)})$'s are strictly affine-invariant.

3.2. A class of angle-based procedures.

The "Tyler" analog of the test statistic (2.2) is

$$\check{Q}_K^{(n)} := \frac{k}{n\,\mathrm{E}[K^2(U)]} \sum_{i,j=1}^{n} K\Big(\frac{\hat{R}_i^{(n)}}{n+1}\Big) K\Big(\frac{\hat{R}_j^{(n)}}{n+1}\Big) (\mathbf{W}_i^{(n)})'\mathbf{W}_j^{(n)}; \tag{3.1}$$

write $\check{Q}_{f_\star}^{(n)}$ for the special case where $K := \varphi_{f_\star} \circ \tilde{F}_{\star k}^{-1}$.

One can easily verify that the proof of (2.3) (see Appendix B in Hallin and Paindaveine 2002a for details) extends to the present situation, yielding

$$\check{Q}_K^{(n)} = \tilde{Q}_{K;\boldsymbol{\Sigma},f}^{(n)} + o_{\mathrm{P}}(1), \tag{3.2}$$

under $\mathcal{H}^{(n)}(\boldsymbol{\theta}_0, \boldsymbol{\Sigma}, f)$. Indeed, that proof only requires the consistency of the estimate of the scalar products (which trivially follows from Lemma 3.1) and the symmetry property

$$(\mathbf{W}_i^{(n)}(\mathbf{s}))'\mathbf{W}_j^{(n)}(\mathbf{s}) = s_i\, s_j\, (\mathbf{W}_i^{(n)})'\mathbf{W}_j^{(n)}, \tag{3.3}$$

for all $\mathbf{s} := (s_1, \ldots, s_n) \in \{-1, 1\}^n$, where $\mathbf{W}_i^{(n)}(\mathbf{s})$ denotes the Tyler residual of $s_i(\mathbf{X}_i^{(n)} - \boldsymbol{\theta}_0)$ computed from the sample $(s_1(\mathbf{X}_1^{(n)} - \boldsymbol{\theta}_0), \ldots, s_n(\mathbf{X}_n^{(n)} - \boldsymbol{\theta}_0))$. The following proposition then is a direct consequence of (3.2) and Proposition 2.2.

Proposition 3.3. *Consider the sequence of tests $\check{\phi}_K^{(n)}$ (resp. $\check{\phi}_{f_\star}^{(n)}$) rejecting the null hypothesis $\mathcal{H}_0^{(n)}$ whenever $\check{Q}_K^{(n)}$ (resp. $\check{Q}_{f_\star}^{(n)}$) exceeds the α-upper quantile $\chi^2_{k,1-\alpha}$ of a chi-square distribution with k degrees of freedom. Then, all statements of Proposition 2.2 hold for $\check{Q}_K^{(n)}$, $\check{Q}_{f_\star}^{(n)}$, $\check{\phi}_K^{(n)}$ and $\check{\phi}_{f_\star}^{(n)}$.*

This proposition shows that angle-based procedures and interdirection-based procedures share the same invariance properties and asymptotic efficiencies. An important advantage of angle-based procedures is that they avoid the computation of interdirections, which is potentially very heavy for large values of the space dimension k.

4. A Monte-Carlo study.

Should interdirection-based procedures then be abandoned in favor of angle-based ones? Do these two asymptotically equivalent classes of procedures enjoy similar small-sample properties? Is one of them more robust against outliers? A Monte-Carlo experiment was conducted in the bivariate case in order to answer these questions.

Letting (without any loss of generality) $\boldsymbol{\theta}_0 = \mathbf{0} \in \mathbb{R}^2$, we generated $N = 2,000$ independent samples $(\mathbf{X}_1, \ldots, \mathbf{X}_{30})$ of size $n = 30$ from eight populations :

($\mathcal{N}$) (elliptical symmetry; light tails) bivariate standard normal $\mathcal{N}_2(\mathbf{0}, \mathbf{I}_2)$,

(t_3) (elliptical symmetry; heavy tails) bivariate standard Student with 3 degrees of freedom,

(Eh) a mixture of $\mathcal{N}_2(\boldsymbol{\mu}, \mathbf{I}_2)$ and $\mathcal{N}_2(-\boldsymbol{\mu}, \mathbf{I}_2)$ with mixing probabilities 1/2, 1/2 and $\boldsymbol{\mu} = (3, 0)'$,

(Ev) a mixture of $\mathcal{N}_2(\boldsymbol{\mu}, \mathbf{I}_2)$ and $\mathcal{N}_2(-\boldsymbol{\mu}, \mathbf{I}_2)$ with mixing probabilities 1/2, 1/2 and $\boldsymbol{\mu} = (0, 3)'$,

(Ec) (*radial* outliers) a mixture of $\mathcal{N}_2(\mathbf{0}, \mathbf{I}_2)$ and $\mathcal{N}_2(\mathbf{0}, \sigma^2\,\mathbf{I}_2)$ with mixing probabilities 0.8, 0.2 and $\sigma = 20$,

(O20) (*angular* outliers) a "deterministic" mixture in which a proportion $(1-c)$ of the sample is drawn from a $\mathcal{N}_2\left(\mathbf{0}, \begin{pmatrix} \sigma^2 & 0 \\ 0 & 1 \end{pmatrix}\right)$ distribution and proportion c is drawn from a mixture of a uniform distribution over the unit ball centered at $\boldsymbol{\mu}$ (probability 1/2), and a uniform distribution over the unit ball centered at $-\boldsymbol{\mu}$ (probability 1/2), with $\sigma = 20$, $c = 0.2$ and $\boldsymbol{\mu} = (0, 2)'$,

(O40) (*angular* outliers) same as (O20), but $c = 0.4$,

($\tilde{\text{O}}$40) (*angular* outliers) same as (O20), but $c = 0.2$ and $\boldsymbol{\mu} = (0, 4)'$.

For each replication, the following tests were performed at nominal probability level $\alpha = 5\%$:

(i) the exact Hotelling test ϕ_{T^2},

(ii) the asymptotic version of Hotelling's test $\phi^a_{T^2}$,

(iii) Randles (1989)'s interdirection-based sign test $\phi^{(n)}_S$ ($K(u) = 1\ \forall u$),

(iv) Peters and Randles (1990)'s interdirection-based Wilcoxon test $\phi^{(n)}_W$ (with score $K(u) = u$),

(v) the interdirection-based van der Waerden test $\phi^{(n)}_{vdW}$; see Hallin and Paindaveine (2002a),

(vi) the interdirection-based test $\phi^{(n)}_{t_3}$ based on bivariate Student scores (three degrees of freedom); see Hallin and Paindaveine (2002a),

(vii) Randles (2000)'s sign test $\check{\phi}^{(n)}_S$ (the angle-based counterpart of $\phi^{(n)}_S$),

(viii) the angle-based counterpart $\check{\phi}^{(n)}_W$ of $\phi^{(n)}_W$,

(ix) the angle-based counterpart $\check{\phi}^{(n)}_{vdW}$ of $\phi^{(n)}_{vdW}$,

(x) the angle-based counterpart $\check{\phi}^{(n)}_{t_3}$ of $\phi^{(n)}_{t_3}$.

Tyler's estimator of scatter was obtained via the iterative scheme described in Randles (2000); iterations were stopped as soon as the Frobenius distance between the left and right hand sides of (2.1) fell below 10^{-8}. The simulation of bivariate Student variables $\mathbf{X}_i$ with 3 degrees of freedom was based on the fact that $\mathbf{X}_i =_{\text{d}} \mathbf{Z}_i/\sqrt{Y_i/3}$, where $\mathbf{Z}_i \sim \mathcal{N}_2(\mathbf{0}, \mathbf{I}_2)$ and $Y_i \sim \chi^2_3$ are independent.

Rejection frequencies, estimating the corresponding sizes and powers, were recorded at four values of $\boldsymbol{\theta}$, of the form $m\boldsymbol{\Delta}$ ($m = 0, 1, 2, 3$), with $\boldsymbol{\Delta} = (.3, 0)'$

for Populations ($\mathcal{N}$), (t_3), (Eh), (Ev) and (Ec), and with $\mathbf{\Delta} = (.3, .3)'$ for Populations (O20), (O40) and ($\tilde{\text{O}}$40); they are reported in Table 1. Note that the standard errors for such estimates (for $N = 2,000$ replications) are .0049, .0089, and .0112 for estimated probabilities (size or power) of the order of $p = .05$ (equivalently, $p = .95$), $p = .20$ (equivalently, $p = .80$), and $p = .50$, respectively.

k=2	(n=30)	Shift			
population	test	0	$\mathbf{\Delta}$	$2\mathbf{\Delta}$	$3\mathbf{\Delta}$
	ϕ_{T^2} / $\phi^a_{T^2}$	.0520 / .0780	.1890 / .2395	.6380 / .7050	.9380 / .9590
	ϕ_S ($\check{\phi}_S$)	.0485 (.0490)	.1535 (.1575)	.5265 (.5270)	.8685 (.8660)
$\mathcal{N}$	ϕ_W ($\check{\phi}_W$)	.0480 (.0450)	.1705 (.1695)	.5770 (.5785)	.9090 (.9090)
	ϕ_{vdW} ($\check{\phi}_{vdW}$)	.0360 (.0405)	.1555 (.1525)	.5495 (.5600)	.9010 (.9000)
	ϕ_{t_3} ($\check{\phi}_{t_3}$)	.0490 (.0475)	.1785 (.1805)	.5745 (.5805)	.8980 (.9050)
	ϕ_{T^2} / $\phi^a_{T^2}$	.0440 / .0705	.1075 / .1445	.3315 / .4055	.6525 / .7035
	ϕ_S ($\check{\phi}_S$)	.0485 (.0490)	.1400 (.1350)	.4400 (.4465)	.7805 (.7740)
t_3	ϕ_W ($\check{\phi}_W$)	.0445 (.0445)	.1145 (.1140)	.3540 (.3580)	.6595 (.6640)
	ϕ_{vdW} ($\check{\phi}_{vdW}$)	.0395 (.0365)	.1055 (.1035)	.3435 (.3465)	.6510 (.6610)
	ϕ_{t_3} ($\check{\phi}_{t_3}$)	.0505 (.0490)	.1530 (.1555)	.4735 (.4830)	.7995 (.8070)
	ϕ_{T^2} / $\phi^a_{T^2}$	.0595 / .0780	.0645 / .0895	.0985 / .1350	.1730 / .2225
	ϕ_S ($\check{\phi}_S$)	.0525 (.0530)	.0530 (.0555)	.0620 (.0610)	.0840 (.0750)
Eh	ϕ_W ($\check{\phi}_W$)	.0615 (.0595)	.0695 (.0665)	.0965 (.0930)	.1605 (.1535)
	ϕ_{vdW} ($\check{\phi}_{vdW}$)	.0555 (.0545)	.0590 (.0575)	.0790 (.0720)	.1210 (.1090)
	ϕ_{t_3} ($\check{\phi}_{t_3}$)	.0615 (.0615)	.0675 (.0710)	.0945 (.0890)	.1575 (.1370)
	ϕ_{T^2} / $\phi^a_{T^2}$	.0545 / .0735	.1990 / .2470	.6400 / .7030	.9350 / .9600
	ϕ_S ($\check{\phi}_S$)	.0495 (.0535)	.1755 (.1790)	.5750 (.5720)	.9115 (.9115)
Ev	ϕ_W ($\check{\phi}_W$)	.0530 (.0545)	.2320 (.2290)	.7085 (.7035)	.9565 (.9545)
	ϕ_{vdW} ($\check{\phi}_{vdW}$)	.0435 (.0445)	.2175 (.2180)	.6960 (.6860)	.9540 (.9520)
	ϕ_{t_3} ($\check{\phi}_{t_3}$)	.0555 (.0600)	.1920 (.1940)	.6080 (.6105)	.9195 (.9215)
	ϕ_{T^2} / $\phi^a_{T^2}$	.0135 / .0295	.0180 / .0355	.0350 / .0605	.0735 / .1100
	ϕ_S ($\check{\phi}_S$)	.0485 (.0490)	.1120 (.1115)	.3545 (.3570)	.6775 (.6805)
Ec	ϕ_W ($\check{\phi}_W$)	.0435 (.0435)	.0985 (.0935)	.2635 (.2715)	.4935 (.4925)
	ϕ_{vdW} ($\check{\phi}_{vdW}$)	.0395 (.0400)	.0855 (.0825)	.2480 (.2465)	.4805 (.4830)
	ϕ_{t_3} ($\check{\phi}_{t_3}$)	.0525 (.0515)	.1335 (.1345)	.4070 (.4090)	.7350 (.7380)
	ϕ_{T^2} / $\phi^a_{T^2}$	.0535 / .0730	.1245 / .1720	.4015 / .4720	.7590 / .8175
	ϕ_S ($\check{\phi}_S$)	.0395 (.0405)	.0815 (.0825)	.2520 (.2370)	.4930 (.4645)
O20	ϕ_W ($\check{\phi}_W$)	.0440 (.0455)	.1105 (.1055)	.3385 (.3200)	.6670 (.6430)
	ϕ_{vdW} ($\check{\phi}_{vdW}$)	.0410 (.0415)	.0965 (.0930)	.3030 (.2770)	.6105 (.5810)
	ϕ_{t_3} ($\check{\phi}_{t_3}$)	.0495 (.0495)	.1025 (.1035)	.3055 (.2980)	.5940 (.5750)
	ϕ_{T^2} / $\phi^a_{T^2}$	.0450 / .0675	.1095 / .1495	.2875 / .3565	.5965 / .6700
	ϕ_S ($\check{\phi}_S$)	.0460 (.0420)	.0850 (.0580)	.1830 (.0920)	.3200 (.1680)
O40	ϕ_W ($\check{\phi}_W$)	.0440 (.0465)	.0925 (.0665)	.2500 (.1345)	.4965 (.2935)
	ϕ_{vdW} ($\check{\phi}_{vdW}$)	.0405 (.0420)	.0755 (.0530)	.1965 (.1040)	.3960 (.2160)
	ϕ_{t_3} ($\check{\phi}_{t_3}$)	.0475 (.0505)	.1145 (.0820)	.2890 (.1730)	.4850 (.3375)
	ϕ_{T^2} / $\phi^a_{T^2}$	.0530 / .0770	.0730 / .1010	.1330 / .1735	.2425 / .3015
	ϕ_S ($\check{\phi}_W$)	.0510 (.0530)	.0875 (.0595)	.1915 (.0945)	.3010 (.1365)
$\tilde{\text{O}}$40	ϕ_W ($\check{\phi}_S$)	.0480 (.0515)	.0920 (.0660)	.2245 (.1185)	.3760 (.1920)
	ϕ_{vdW} ($\check{\phi}_{vdW}$)	.0410 (.0460)	.0800 (.0575)	.1935 (.0985)	.3285 (.1475)
	ϕ_{t_3} ($\check{\phi}_{t_3}$)	.0510 (.0565)	.1000 (.0715)	.2215 (.1110)	.3590 (.1725)

Table 1 Estimated sizes and powers of the Hotelling test, and of the interdirection- and angle-based tests with sign scores, Wilcoxon scores, van der Waerden scores and t_3-scores, under various values of the shift and various densities; simulations are based on 2,000 bivariate samples of size 30.

In Table 2, we report the rejection frequencies, corresponding to the same populations and tests, for sample size $n = 300$. The shifts are still of the form $m\mathbf{\Delta}$

($m = 0, 1, 2, 3$), with $\mathbf{\Delta} = (.07, 0)'$ for Populations ($\mathcal{N}$), (t_3), (Eh), (Ev) and (Ec), and with $\mathbf{\Delta} = (.07, .07)'$ for Populations (O20), (O40) and (Õ40).

k=2	(n=300)	Shift			
population	test	0	$\mathbf{\Delta}$	$2\mathbf{\Delta}$	$3\mathbf{\Delta}$
	ϕ_{T^2} / $\phi^a_{T^2}$	.0455 / .0475	.1805 / .1845	.5715 / .5810	.9140 / .9170
	ϕ_S ($\check{\phi}_S$)	.0495 (.0510)	.1575 (.1590)	.4780 (.4740)	.8260 (.8280)
$\mathcal{N}$	ϕ_W ($\check{\phi}_W$)	.0475 (.0480)	.1705 (.1715)	.5620 (.5595)	.9125 (.9125)
	ϕ_{vdW} ($\check{\phi}_{vdW}$)	.0425 (.0455)	.1745 (.1780)	.5655 (.5605)	.9135 (.9115)
	ϕ_{t_3} ($\check{\phi}_{t_3}$)	.0485 (.0490)	.1575 (.1575)	.4960 (.4920)	.8465 (.8475)
	ϕ_{T^2} / $\phi^a_{T^2}$	.0440 / .0485	.0960 / .0990	.2555 / .2600	.5200 / .5265
	ϕ_S ($\check{\phi}_S$)	.0495 (.0510)	.1350 (.1360)	.4175 (.4180)	.7635 (.7620)
t_3	ϕ_W ($\check{\phi}_W$)	.0490 (.0500)	.1170 (.1215)	.3520 (.3530)	.6905 (.6880)
	ϕ_{vdW} ($\check{\phi}_{vdW}$)	.0485 (.0495)	.1215 (.1210)	.3525 (.3515)	.6790 (.6785)
	ϕ_{t_3} ($\check{\phi}_{t_3}$)	.0505 (.0495)	.1395 (.1375)	.4400 (.4380)	.7840 (.7850)
	ϕ_{T^2} / $\phi^a_{T^2}$	.0440 / .0480	.0540 / .0575	.0960 / .0995	.1570 / .1590
	ϕ_S ($\check{\phi}_S$)	.0495 (.0480)	.0510 (.0520)	.0610 (.0570)	.0800 (.0685)
Eh	ϕ_W ($\check{\phi}_W$)	.0590 (.0575)	.0670 (.0670)	.1035 (.0915)	.1775 (.1510)
	ϕ_{vdW} ($\check{\phi}_{vdW}$)	.0570 (.0595)	.0650 (.0630)	.0910 (.0820)	.1410 (.1230)
	ϕ_{t_3} ($\check{\phi}_{t_3}$)	.0475 (.0450)	.0570 (.0560)	.1050 (.0890)	.1740 (.1445)
	ϕ_{T^2} / $\phi^a_{T^2}$	.0455 / .0475	.1765 / .1805	.5805 / .5890	.9195 / .9225
	ϕ_S ($\check{\phi}_S$)	.0440 (.0415)	.1575 (.1580)	.5130 (.5065)	.8710 (.8665)
Ev	ϕ_W ($\check{\phi}_W$)	.0510 (.0475)	.2350 (.2280)	.6745 (.6675)	.9540 (.9510)
	ϕ_{vdW} ($\check{\phi}_{vdW}$)	.0505 (.0460)	.2365 (.2305)	.6835 (.6765)	.9550 (.9530)
	ϕ_{t_3} ($\check{\phi}_{t_3}$)	.0490 (.0455)	.1580 (.1570)	.5085 (.5040)	.8560 (.8530)
	ϕ_{T^2} / $\phi^a_{T^2}$	.0475 / .0500	.0510 / .0525	.0555 / .0575	.0655 / .0685
	ϕ_S ($\check{\phi}_S$)	.0495 (.0510)	.1185 (.1195)	.3305 (.3320)	.6460 (.6455)
Ec	ϕ_W ($\check{\phi}_W$)	.0520 (.0520)	.1060 (.1060)	.2675 (.2725)	.5545 (.5555)
	ϕ_{vdW} ($\check{\phi}_{vdW}$)	.0510 (.0520)	.1035 (.1050)	.2520 (.2520)	.5305 (.5285)
	ϕ_{t_3} ($\check{\phi}_{t_3}$)	.0470 (.0470)	.1310 (.1300)	.3840 (.3840)	.7125 (.7110)
	ϕ_{T^2} / $\phi^a_{T^2}$	.0440 / .0475	.1175 / .1230	.3800 / .3860	.7295 / .7370
	ϕ_S ($\check{\phi}_S$)	.0490 (.0455)	.1000 (.0905)	.2615 (.2455)	.5450 (.4955)
O20	ϕ_W ($\check{\phi}_W$)	.0515 (.0490)	.1195 (.1095)	.3775 (.3460)	.7325 (.6970)
	ϕ_{vdW} ($\check{\phi}_{vdW}$)	.0505 (.0480)	.1135 (.1055)	.3560 (.3335)	.7175 (.6735)
	ϕ_{t_3} ($\check{\phi}_{t_3}$)	.0515 (.0465)	.1015 (.1020)	.2810 (.2600)	.5635 (.5155)
	ϕ_{T^2} / $\phi^a_{T^2}$	.0490 / .0500	.1055 / .1100	.2840 / .2925	.5755 / .5805
	ϕ_S ($\check{\phi}_S$)	.0475 (.0520)	.0870 (.0625)	.2045 (.0915)	.3990 (.1535)
O40	ϕ_W ($\check{\phi}_W$)	.0610 (.0660)	.1055 (.0925)	.3200 (.1910)	.6440 (.4180)
	ϕ_{vdW} ($\check{\phi}_{vdW}$)	.0625 (.0670)	.0980 (.0825)	.2700 (.1530)	.5415 (.3020)
	ϕ_{t_3} ($\check{\phi}_{t_3}$)	.0530 (.0525)	.1100 (.0790)	.3115 (.1765)	.5870 (.3445)
	ϕ_{T^2} / $\phi^a_{T^2}$	.0470 / .0495	.0630 / .0645	.1165 / .1215	.2195 / .2245
	ϕ_S ($\check{\phi}_S$)	.0525 (.0550)	.0950 (.0640)	.2175 (.1020)	.3865 (.1525)
Õ40	ϕ_W ($\check{\phi}_W$)	.0505 (.0500)	.0960 (.0685)	.2650 (.1405)	.5380 (.2720)
	ϕ_{vdW} ($\check{\phi}_{vdW}$)	.0505 (.0485)	.0915 (.0650)	.2330 (.1150)	.4790 (.2170)
	ϕ_{t_3} ($\check{\phi}_{t_3}$)	.0480 (.0525)	.0935 (.0630)	.2310 (.1000)	.4380 (.1560)

Table 2 Estimated sizes and powers of the Hotelling test, and of the interdirection- and angle-based tests with sign scores, Wilcoxon scores, van der Waerden scores and t_3-scores, under various values of the shift and various densities; simulations are based on 2,000 bivariate samples of size 300.

These simulations indicate that both classes of procedures behave quite similarly under the two elliptically symmetric densities (($\mathcal{N}$) and (t_3)) considered. Angle-based procedures for small sample sizes appear to be slightly more powerful; this superiority however is hardly significant. Note that, as expected, the Hotelling tests (resp., ϕ_{t_3} and $\check{\phi}_{t_3}$) are best under Gaussian densities (resp., under

t_3 densities); the Hotelling tests however suffer serious type I error problems. Except for the (too) small sample size 30, the optimality under Gaussian densities of the van der Waerden tests ϕ_{vdW} and $\check{\phi}_{vdW}$ is confirmed. Under (t_3), these tests dominate the Hotelling test—a numerical illustration of the extended Chernoff-Savage result proved in Hallin and Paindaveine (2002a).

For the mixtures (Eh) and (Ev), interdirection-based procedures appear to be slightly better than their angle-based counterparts. The mixture (Ec) can be considered as a standard normal distribution, with a proportion 0.2 of *radial* outliers. Since interdirection-based and angle-based procedures differ from each other only through the *angular* part, they roughly suffer the same losses of power due to the presence of radial outliers. Their relative performances thus remain the same as under strictly elliptical distributions.

Unlike radial outliers, *angular* outliers clearly quite substantially affect both classes of procedures. For both small and large sample sizes, interdirection-based procedures appear to be very significantly more robust than the angle-based ones. When the proportion of outliers is as high as 0.4 (see (O40)), the lack of power of angle-based procedures is dramatic. The same conclusion holds in the presence of "more extreme" angular outliers (see ($\tilde{\text{O}}$40)).

A similar Monte-Carlo study was also performed in the trivariate case for the same types of distributions. This simulation leads to the same conclusions as in the bivariate case.

5. Conclusions and final comments.

In this paper, we introduce a class of angle-based procedures that are asymptotically equivalent to interdirection-based procedures developed in Hallin and Paindaveine (2002a). These procedures are computationally preferable, since they avoid the computation of interdirections. Their small-sample performance is not significantly different from that of their interdirection-based counterparts under strict ellipticity, or in the presence of *radial outliers.* But interdirection-based procedures exhibit much better resistance in the presence of *angular* outliers.

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