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Existence and breakdown analysis of M-quantiles in general Hilbert spaces

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Abstract: M-quantiles extend M-estimators of location in the same way the usual quantiles extend the median. Since this extension is motivated by the desire to achieve a trade-off between robustness and efficiency, it is surprising that a robustness analysis of M-quantiles remains unavailable to date. Such an analysis is missing even for univariate M-quantiles, hence also for their most common multivariate extension, namely *spatial* or *geometric* M-quantiles. In this paper, we therefore study the global robustness of M-quantiles in terms of breakdown point. We do so in a general framework where M-quantiles, to the best of our knowledge, have not been considered earlier, namely in possibly infinite-dimensional Hilbert spaces (which covers the case of functional M-quantiles). Existence of M-quantiles in such a general framework remains an open question, though, and we thus first establish existence through weak topology arguments. Then, we study the breakdown point of M-quantiles. We provide an account of this question that requires only very mild assumptions on the convex loss function at hand. As a result, our analysis is considerably more general than the one, almost exclusively conducted for quantiles, in [20]. Such generality requires original results on regular variation of convex loss functions. In order to handle the possible non-uniqueness of M-quantiles, we also need to consider lower and upper breakdown point concepts.

Keywords and phrases: Breakdown point, expectiles, functional data, M-quantiles, multivariate quantiles, robustness.

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1. Introduction

For any $\tau \in (0,1)$, the usual τ-quantiles of real-valued observations $x_1, \ldots, x_n$ are defined as the minimizers of the asymmetric objective function

$$\mu(\in \mathbb{R}) \mapsto M_{\tau,\rho}(\mu; x_1, \ldots, x_n) \tag{1.1}$$

$$= \frac{2}{n} \sum_{i=1}^{n} \Big(\tau \rho(|x_i - \mu|) \mathbb{I}[x_i - \mu > 0] + (1-\tau)\rho(|x_i - \mu|)\mathbb{I}[x_i - \mu < 0] \Big)$$

$$= \frac{1}{n} \sum_{i=1}^{n} \rho(|x_i - \mu|) \bigg(1 + \frac{(2\tau - 1)(x_i - \mu)}{|x_i - \mu|} \mathbb{I}[x_i \neq \mu] \bigg) \tag{1.2}$$

with $\rho(t) = t$; throughout, $\mathbb{I}[A]$ will denote the indicator function associated with condition A. Choosing $\rho(t) = t^2$ and $\rho(t) = t^p$ for $p \geq 1$ rather provides expectiles ([25]) and L_p-quantiles

([8]), respectively, whereas, more generally, a generic convex function $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\rho(0) = 0$ leads to M-quantiles ([3]). By considering an arbitrary value of $\tau \in (0, 1)$ rather than $\tau = 1/2$, expectiles (resp., M-quantiles) extend the mean (resp., M-estimators of location) in the same way the usual quantiles extend the median. Expectiles and M-quantiles have recently found important applications, in particular in risk analysis, tail inference, extreme value theory [10]–[11]. In small area estimation (SAE), M-quantiles provide outlier-robust predictors of domain means, quantiles and poverty indicators, together with nonparametric bootstrap MSE estimators [5, 22, 30]. Also, for clustered and survey data, M-quantile random-effects regression extends inference to multilevel designs and allows incorporation of sampling weights [27, 28, 29].

If $x_1, \ldots, x_n$ are observations in a (possibly infinite-dimensional) Hilbert space H, with inner product $\langle \cdot, \cdot \rangle$ and corresponding norm $\| \cdot \|$, M-quantiles of order $\alpha \in [0, 1)$ in direction $u \in \mathcal{S}_H = \{x \in H : \|x\| = 1\}$ are defined as the minimizers of the objective function

$$\mu(\in H) \mapsto M^{\rho}_{\alpha,u}(\mu; x_1, \ldots, x_n) = \frac{1}{n} \sum_{i=1}^{n} \rho(\|x_i - \mu\|) \left(1 + \frac{\alpha \, \langle u, x_i - \mu \rangle}{\|x_i - \mu\|} \mathbb{I}[x_i \neq \mu] \right). \tag{1.3}$$

For $H = \mathbb{R}$, the minimizers of (1.3) provide the M-quantiles defined above in a *center-outward parametrization* where the M-quantile of order τ is described as the M-quantile of order α in direction $u \in \{-1, 1\}$, with $\alpha u = 2\tau - 1$. In this alternative parametrization, which is standard when considering multivariate quantiles (see, e.g., [9], [15], [18]), $\alpha = 0$ provides the innermost quantile (irrespective of the direction u) and increasing values of α provide more and more extreme quantiles in direction u. In $\mathbb{R}^d$, taking $\rho(t) = t$ provides the celebrated *spatial* or *geometric* quantiles from [6] and [14], whereas $\rho(t) = t^2$ yields the geometric expectiles from [17]. For a generic loss function ρ, the corresponding M-quantiles were recently studied in [18]; see also [3]. All these contributions focus on $\mathbb{R}^d$, and, to the best of our knowledge, the present paper is the first one to consider M-quantiles in general Hilbert spaces (only standard spatial quantiles have been considered earlier in infinite-dimensional spaces; see [4]).

Quite naturally, M-quantiles were originally introduced to achieve a trade-off between robustness and efficiency, just like their innermost version, namely M-estimators of location, aimed at inheriting some of the efficiency of the sample mean and robustness of the sample median. In this light, it is quite surprising that most formal results on the qualitative robustness of M-quantiles—measured through the classical breakdown point concept from [13] and [16]—concern the usual quantiles associated with $\rho(t) = t$; see [20][1]. In particular, no results are available for the breakdown point of M-quantiles based on the Huber loss function defined by $\rho_c(t) = (t^2/2)\mathbb{I}[0 < t < c] + c(t - (c/2))\mathbb{I}[t \geq c]$, with $c > 0$, which are the M-quantiles one would classically consider to achieve a trade-off between efficiency and robustness.

The goal of the present work is then twofold:

(a) we aim to fill the gap above by providing an almost complete account of the breakdown point properties of M-quantiles, be they in $\mathbb{R}$, $\mathbb{R}^d$ or in infinite-dimensional Hilbert spaces. Our main emphasis is to show how these properties depend on the corresponding

[1]We also refer to [1] and [26] for recent papers that derive the breakdown point of competing concepts of multivariate quantiles, that are based on optimal transport.

loss function ρ. In particular, we show that if the loss function is sub-linear, in the sense that there exists $C > 0$ such that $\rho(t) \leq Ct$ for all $t > 0$, then the breakdown point of the resulting M-quantiles is that of the geometric quantiles; otherwise, M-quantiles display no robustness at all and their breakdown point is $1/n$.

(b) Theorem 2.1 from [18] establishes existence of M-quantiles[2] when $H = \mathbb{R}^d$. When H is an infinite-dimensional space, however, existence is a much more delicate—in fact, open—question. We therefore aim to establish existence of M-quantiles in general Hilbert spaces, which will require weak topology arguments.

The outline of the paper is as follows: in Section 2, we establish existence of M-quantiles in general, hence possibly infinite-dimensional, Hilbert spaces. In Section 3, we define the concepts of lower and upper breakdown points (BDPs) we adopt in this work (both versions of BDPs are needed to cope with the possible non-uniqueness of M-quantiles). In Section 4, we obtain original results on the regular variation of convex loss functions that will play a key role in our BDP analysis. In Section 5, we derive an upper bound on the upper BDP of M-quantiles. This upper bound crucially depends on the loss function ρ and may be so low for some loss functions that it allows us to deduce the BDPs of the corresponding M-quantiles. In Section 6, we obtain a lower bound on the lower BDP, which, in most cases, provides the BDPs of M-quantiles for a generic loss function ρ. In Section 7, we conclude by providing some final comments. Two appendices prove auxiliary results and provide additional details.

2. Existence of M-quantiles in general Hilbert spaces

Throughout, we consider loss functions ρ that belong to the class $\mathcal{C}$ collecting the functions $\rho : [0,\infty) \to [0,\infty)$ that are convex, satisfy $\rho(t) = 0$ for $t = 0$ only, and are twice continuously differentiable on $(0,\infty) \setminus \mathcal{N}$, where $\mathcal{N} \subset (0,\infty)$ is at most countable and has no accumulation point in $[0,\infty)$. We then denote as $\mathcal{P}_H^\rho$ the class of probability measures P on H such that for any $\mu \in H$ there exists $\delta_\mu > 0$ for which

$$\int_H \psi_-(\|z-\mu\| + \delta_\mu)\, dP(z) < \infty, \tag{2.4}$$

where ψ_- stands for the left-derivative of ρ (existence of ψ_- follows from the convexity of ρ). Observe that for the power loss functions defined by $\rho(t) = t^p$ with $p \geq 1$, the condition $P \in \mathcal{P}_H^\rho$ is equivalent to $\int_H \|z\|^{p-1}\, dP(z) < \infty$, *i.e.*, P has finite moments of order $p-1$.

We start by defining the M-quantiles in more generality, for an arbitrary probability measure $P \in \mathcal{P}_H^\rho$, similarly to Definition 1 in [18] (recall that H may here be an infinite-dimensional Hilbert space).

Definition 2.1. *Let $\rho \in \mathcal{C}$ and $P \in \mathcal{P}_H^\rho$. Fix $\alpha \in [0,1)$ and $u \in \mathcal{S}_H$. We say that $\mu_{\alpha,u}^\rho$ is an M-quantile of order α in direction u for P if and only if it minimizes the objective function*

$$\mu \mapsto M_{\alpha,u}^\rho(\mu) := \int_H \left\{H_{\alpha,u}^\rho(z-\mu) - H_{\alpha,u}^\rho(z)\right\} dP(z) \tag{2.5}$$

[2]The terminology *ρ-quantiles* is adopted in [18], but here we will rather speak of *M-quantiles*, which is more common in the literature.

over H, where we let

$$H^{\rho}_{\alpha,u}(z) := \rho(\|z\|)\left(1 + \alpha\frac{\langle u, z\rangle}{\|z\|}\xi_{z,0}\right),$$

with $\xi_{z_1,z_2} = \mathbb{I}[z_1 \neq z_2]$.

The next result follows from Theorem 2.1 in [18] and Lemma S.2.1 in [19] by noticing that the proof transposes *mutatis mutandis* to the infinite-dimensional case.

Proposition 2.1. *Let* $\rho \in \mathcal{C}$ *and* $P \in \mathcal{P}^{\rho}_{H}$. *Fix* $\alpha \in [0, 1)$ *and* $u \in \mathcal{S}_H$. *Then, (i) for any* $\mu \in H$, $M^{\rho}_{\alpha,u}(\mu)$ *is well-defined (i.e., the integral in (2.5) exists and is finite); (ii)* $M^{\rho}_{\alpha,u}$ *is continuous over H; (iii)* $M^{\rho}_{\alpha,u}$ *is coercive in the sense that*[3]

$$\liminf_{\ell\to\infty} \frac{M^{\rho}_{\alpha,u}(\mu_\ell)}{\|\mu_\ell\|} > 0 \tag{2.6}$$

for any sequence (μ_ℓ) *in H such that* $\|\mu_\ell\| \to \infty$.

Existence of M-quantiles in the Euclidean case, established in Theorem 2.1 of [18], easily follows from Proposition 2.1 by a standard ‘coercivity + continuity’ argument: coercivity allows us to reduce the minimization problem in Definition 2.1 to a closed ball, and continuity ensures that a minimum over this (compact) ball exists. When H is infinite-dimensional, however, this reasoning fails since closed balls are no longer compact. As we will show in Theorem 2.1 below, M-quantiles still exist in such a general framework.

Establishing existence will require exploiting the weak lower-semicontinuity of $M^{\rho}_{\alpha,u}$ and the weak compactness of closed balls. Let us recall some standard concepts of weak topology in Hilbert spaces. A sequence (μ_k) in H is said to converge weakly to $\mu \in H$, and we write $\mu_k \rightharpoonup \mu$, if $\langle \mu_k, h\rangle \to \langle \mu, h\rangle$ for all $h \in H$. Strong convergence is convergence in the topology of H: $\|\mu_k - \mu\| \to 0$. Strong convergence implies weak convergence, but the converse is false when H is infinite-dimensional. A map $T : H \to \mathbb{R}$ is called weakly lower-semicontinuous if

$$T(\mu) \leq \liminf_{k\to\infty} T(\mu_k)$$

for all $\mu \in H$ and any sequence (μ_k) converging weakly to μ. Weak lower-semicontinuity is thus a stronger requirement than lower-semicontinuity since the inequality in the previous display must hold for any *weakly* converging sequence (and not only for any strongly converging sequence). Finally, a subset B of H is said to be weakly compact if any sequence (μ_k) in B admits a subsequence converging weakly in B. In Hilbert spaces, closed balls are weakly compact; see, e.g., Theorem 6.7.3 in [2].

When the objective function $M^{\rho}_{\alpha,u}$ is convex, it is weakly lower-semicontinuous. Even in the Euclidean case $H = \mathbb{R}^d$, however, Theorem 3.1 in [18] entails that $M^{\rho}_{\alpha,u}$ is convex for any $\alpha \in [0, 1)$ and $u \in \mathcal{S}_H$ if and only if $t \mapsto t^2/\rho(t)$ is concave; for the power loss functions $\rho(t) = t^p$ with $p \geq 1$, this condition holds if and only if $p \in [1, 2]$, which is quite a stringent restriction. Worse: for Huber loss functions, this condition does not hold, and $\alpha = 0$ is actually the *only* value of $\alpha \in [0, 1)$ for which $M^{\rho}_{\alpha,u}$ is convex for all $u \in \mathcal{S}^{d-1}$; see the comments following Theorem 3.2 in [18]. Quite fortunately, convexity of $M^{\rho}_{\alpha,u}$ is not needed for weak lower-semicontinuity, as the following result shows.

[3]The convergence in (2.6) guarantees that $M^{\rho}_{\alpha,u}(\mu_\ell)$ diverges to infinity at least linearly with $\|\mu_\ell\|$. This is a stronger statement than the usual coercivity that only requires that $M^{\rho}_{\alpha,u}(\mu_\ell)$ diverges to infinity as $\|\mu_\ell\|$ does.

Lemma 2.1. *Let $\rho \in \mathcal{C}$ and $P \in \mathcal{P}_H^\rho$. Then, for any $\alpha \in [0,1)$ and $u \in \mathcal{S}_H$, the map $\mu \mapsto M_{\alpha,u}^\rho(\mu)$ is weakly lower-semicontinuous.*

Proof. Let $\mu \in H$ and let (μ_k) be a sequence in H such that $\mu_k \rightharpoonup \mu$. In view of Lemma A.3, Fatou's lemma entails that

$$\begin{aligned}\liminf_{k\to\infty} M_{\alpha,u}^\rho(\mu_k) &= \liminf_{k\to\infty} \int_H \left\{H_{\alpha,u}^\rho(z-\mu_k) - H_{\alpha,u}^\rho(z)\right\} dP(z) \\ &\geq \int_H \liminf_{k\to\infty} \left\{H_{\alpha,u}^\rho(z-\mu_k) - H_{\alpha,u}^\rho(z)\right\} dP(z). \qquad (2.7)\end{aligned}$$

Defining $\varphi(t) := (\rho(t)/t)\mathbb{I}[t \neq 0] + c_\rho \mathbb{I}[t=0]$, where $c_\rho := \lim_{t\to 0}(\rho(t)/t)$ exists since the convexity of ρ and the fact that $\rho(0)=0$ entail that $t \mapsto (\rho(t)-\rho(0))/(t-0) = \rho(t)/t$ is monotone non-decreasing, note that

$$H_{\alpha,u}^\rho(z) = \varphi(\|z\|)(\|z\| + \alpha\langle u, z\rangle)$$

for all $z \in H$. Since $\mu_k \rightharpoonup \mu$, we have

$$\begin{aligned}\liminf_{k\to\infty} \|\mu_k\|^2 &= \liminf_{k\to\infty} \left(\|\mu_k-\mu\|^2 + 2\langle \mu_k-\mu, \mu\rangle + \|\mu\|^2\right) \\ &\geq \liminf_{k\to\infty} \left(2\langle \mu_k, \mu\rangle - \|\mu\|^2\right) \\ &= \|\mu\|^2,\end{aligned}$$

hence also

$$\begin{aligned}\liminf_{k\to\infty} \|z-\mu_k\|^2 &= \liminf_{k\to\infty} \left(\|z\|^2 - 2\langle z, \mu_k\rangle + \|\mu_k\|^2\right) \\ &\geq \|z\|^2 - 2\langle z, \mu\rangle + \|\mu\|^2 \\ &= \|z-\mu\|^2\end{aligned}$$

for all $z \in H$. Since ρ is convex and $\rho(0)=0$, it is straightforward to see that φ is continuous and non-decreasing over $[0,\infty)$. Therefore, $t \mapsto \varphi(\sqrt{t})$ is continuous and non-decreasing over $[0,\infty)$, too, which implies that

$$\liminf_{k\to\infty} \varphi(\|z-\mu_k\|) = \liminf_{k\to\infty} \varphi(\sqrt{\|z-\mu_k\|^2}) \geq \varphi(\sqrt{\|z-\mu\|^2}) = \varphi(\|z-\mu\|)$$

for all $z \in H$. Similarly, we also have

$$\liminf_{k\to\infty} \left(\|z-\mu_k\| + \alpha\langle u, z-\mu_k\rangle\right) \geq \|z-\mu\| + \alpha\langle u, z-\mu\rangle$$

for all $z \in H$. By nonnegativity, we deduce that

$$\begin{aligned}\liminf_{k\to\infty} H_{\alpha,u}^\rho(z-\mu_k) &= \liminf_{k\to\infty} \varphi(\|z-\mu_k\|)\left(\|z-\mu_k\| + \alpha\langle u, z-\mu_k\rangle\right) \\ &\geq \liminf_{k\to\infty} \varphi(\|z-\mu_k\|) \times \liminf_{k\to\infty} \left(\|z-\mu_k\| + \alpha\langle u, z-\mu_k\rangle\right) \\ &\geq \varphi(\|z-\mu\|)\left(\|z-\mu\| + \alpha\langle u, z-\mu\rangle\right) \\ &= H_{\alpha,u}^\rho(z-\mu)\end{aligned}$$

for all $z \in H$. Combining this with (2.7) provides

$$\begin{aligned}\liminf_{k\to\infty} M^{\rho}_{\alpha,u}(\mu_k) &\geq \int_H \liminf_{k\to\infty} \left\{H^{\rho}_{\alpha,u}(z-\mu_k) - H^{\rho}_{\alpha,u}(z)\right\} dP(z) \\ &= \int_H \left\{\liminf_{k\to\infty} H^{\rho}_{\alpha,u}(z-\mu_k) - H^{\rho}_{\alpha,u}(z)\right\} dP(z) \\ &\geq \int_H \left\{H^{\rho}_{\alpha,u}(z-\mu) - H^{\rho}_{\alpha,u}(z)\right\} dP(z) = M^{\rho}_{\alpha,u}(\mu),\end{aligned}$$

which concludes the proof. □

Combined with the weak compactness of closed balls, the weak lower-semicontinuity result above allows us to adapt the 'coercivity + continuity' argument that is used in Euclidean spaces to establish existence of M-quantiles in general Hilbert spaces. We have the following result.

Theorem 2.1. *Let $\rho \in \mathcal{C}$ and $P \in \mathcal{P}^{\rho}_{H}$. Fix $\alpha \in [0,1)$ and $u \in \mathcal{S}_H$. Then, (i) P admits an M-quantile $\mu^{\rho}_{\alpha,u}$ of order α in direction u. (ii) Further assume that $t \mapsto t^2/\rho(t)$ is concave on $(0,\infty)$. If there is no open interval in $(0,\infty)$ on which ψ_- is constant or if P is not supported on a single line of H, then $\mu^{\rho}_{\alpha,u}$ is unique.*

Proof. (i) Denote as B_R the closed ball of H centered at the origin with radius R. Proposition 2.1(iii) implies that there exists $R > 0$ such that $M^{\rho}_{\alpha,u}(\mu) > M^{\rho}_{\alpha,u}(0)$ for all μ in the complement of B_R, so that minimizing $M^{\rho}_{\alpha,u}$ over H is equivalent to minimizing it over B_R only. Let (μ_k) be a sequence in B_R such that

$$M^{\rho}_{\alpha,u}(\mu_k) \to \inf\{M^{\rho}_{\alpha,u}(\mu) : \mu \in B_R\}.$$

Since B_R is weakly compact, we have that, up to extraction of a subsequence, (μ_k) converges weakly in B_R. Denote the limit by μ_0. Weak lower-semicontinuity of $M^{\rho}_{\alpha,u}$ (Lemma 2.1) then entails that

$$M^{\rho}_{\alpha,u}(\mu_0) \leq \lim_{k\to\infty} M^{\rho}_{\alpha,u}(\mu_k) = \inf\{M^{\rho}_{\alpha,u}(\mu) : \mu \in B_R\}.$$

This implies that μ_0 is a minimum of $M^{\rho}_{\alpha,u}$ over B_R, hence is a minimum of $M^{\rho}_{\alpha,u}$ over H. Therefore, μ_0 is an M-quantile of order α in direction u for P.

(ii) Uniqueness of $\mu^{\rho}_{\alpha,u}$ will follow if, under our assumptions, we can prove that $M^{\rho}_{\alpha,u}$ is strictly convex on H. Proving this, in turn, follows exactly as in the proof of Theorem 3.3 in [18], and relies on a number of preliminary results in the same reference that all apply when $\mathbb{R}^d$ is replaced by H and that we enumerate for the sake of completeness: Lemma 3.1, Lemma S.5.1 that allows one to prove Lemma S.3.1, which, combined with Theorem 3.1 and Lemma S.3.3[4], leads to the proof of Theorem 3.3. □

A striking consequence of Theorem 2.1 is that, similarly to the Euclidean case, M-quantiles of any $P \in \mathcal{P}^{\rho}_{H}$ for any order $\alpha \in [0,1)$ and direction $u \in \mathcal{S}_H$ exist and are unique when ρ is the power loss function $\rho(t) = t^p$ with $p \in (1,2]$, *even when P is concentrated on a single line of H*. In particular, any probability measure on H with finite first moments has unique functional (Hilbertian) expectiles of any order $\alpha \in [0,1)$ in any direction $u \in \mathcal{S}_H$.

[4]Notice how Lemma S.3.3 crucially relies on the uniform convexity of Hilbert spaces; this provides meaningful insights as to why the results presented here do not all extend to arbitrary Banach spaces.

While this section settles the question of existence of M-quantiles, we point out that practical computation is not a trivial issue. As already mentioned, when $H = \mathbb{R}^d$ the objective function $M^\rho_{\alpha,u}$ is convex for any $\alpha \in [0,1)$ and $u \in \mathcal{S}_H$ only if $t \mapsto t^2/\rho(t)$ is concave over $(0,\infty)$ (*e.g.*, $\rho(t) = t^p$ with $p \in [1,2]$). In such convex cases, both in $\mathbb{R}^d$ and in infinite-dimensional settings, after projection onto a finite-dimensional subspace of H (*i.e.*, restrict to finite-dimensional subspaces $H_m \subset H$ with dense union in a sieve/Galerkin sense; see [7]), M-quantiles can be obtained via standard convex optimization. Outside this framework, however, the objective may be non-convex, so global optimization is more delicate; a general-purpose algorithmic theory with guarantees in infinite-dimensional Hilbert spaces remains open.

3. Breakdown point of M-quantiles

When the M-quantile $\mu^\rho_{\alpha,u}$ associated with a sample $x_1,\dots,x_n$ is not unique, we *cannot* define the breakdown point of $\mu^\rho_{\alpha,u}$ as the breakdown point of the barycenter of the set of all M-quantiles of order α in direction u, as was done in [20]. Indeed, the objective function in (1.3) may be non-convex when $t \mapsto t^2/\rho(t)$ is not concave over $(0,\infty)$, so that the set of M-quantiles $\mu^\rho_{\alpha,u}(x_1,\dots,x_n)$ may be non-convex, too; in such a case the barycenter may fail to be an M-quantile itself. As already mentioned, for $H = \mathbb{R}^d$ and the Huber loss function $\rho_c(t) = (t^2/2)\mathbb{I}[0 < t < c] + c(t - c/2)\mathbb{I}[t \geq c]$, with $c > 0$, the only value of $\alpha \in [0,1)$ for which the objective function in (1.3) is convex in all directions $u \in \mathcal{S}^{d-1}$ is $\alpha = 0$.

In this context, two natural concepts of breakdown point arise. For a given order $\alpha \in [0,1)$ and direction $u \in \mathcal{S}_H$, consider first the smallest fraction of the sample one needs to perturb to be able to send *all* corresponding M-quantiles outside any bounded subset of H. This leads to the *upper breakdown point*

$$\overline{\mathrm{BDP}}(\mu^\rho_{\alpha,u}; x_1,\dots,x_n) = \min_{\ell\in\{1,\dots,n\}} \left\{ \frac{\ell}{n} : \sup_y \inf_{\mu^\rho_{\alpha,u}} \|\mu^\rho_{\alpha,u}(x_1,\dots,x_n) - \mu^\rho_{\alpha,u}(y_1,\dots,y_n)\| = \infty \right\},$$

where the supremum is taken over all samples $y_1,\dots,y_n$ in H that differ from $x_1,\dots,x_n$ by at most ℓ observations, and where the infimum ranges over all M-quantiles of order α in direction u associated with the sample $y_1,\dots,y_n$ (the infimum is thus relevant only when these quantiles are not unique). Alternatively, consider the smallest fraction of the sample one needs to perturb to be able to send *at least one* corresponding M-quantile outside any bounded subset of H, leading to the *lower breakdown point*

$$\underline{\mathrm{BDP}}(\mu^\rho_{\alpha,u}; x_1,\dots,x_n) = \min_{\ell\in\{1,\dots,n\}} \left\{ \frac{\ell}{n} : \sup_y \sup_{\mu^\rho_{\alpha,u}} \|\mu^\rho_{\alpha,u}(x_1,\dots,x_n) - \mu^\rho_{\alpha,u}(y_1,\dots,y_n)\| = \infty \right\},$$

where the first supremum is taken over all samples $y_1,\dots,y_n$ in H that differ from $x_1,\dots,x_n$ by at most ℓ observations, and the second supremum ranges over all M-quantiles of order α in direction u associated with the sample $y_1,\dots,y_n$. Obviously, we always have

$$\underline{\mathrm{BDP}}(\mu^\rho_{\alpha,u}; x_1,\dots,x_n) \leq \overline{\mathrm{BDP}}(\mu^\rho_{\alpha,u}; x_1,\dots,x_n).$$

If $\mu^{\rho}_{\alpha,u}(x_1,\dots,x_n)$ is unique, then the equality clearly holds in the previous display. In Sections 5–6, we will obtain upper and lower bounds for both breakdown point concepts introduced in the present section. We will do so in a quite satisfactory way since we will obtain a lower bound for $\underline{\mathrm{BDP}}$ and an upper bound for $\overline{\mathrm{BDP}}$, and we will see that these bounds almost always match (moreover, when the bounds do not match, they differ only by the smallest possible positive quantity, namely $1/n$).

4. Regular variation of convex loss functions

Part of our results for the BDP of M-quantiles will require understanding the asymptotic properties—in a regular variation sense—of the loss functions $\rho \in C$. Recall that a measurable function $H : \mathbb{R}^+ \to \mathbb{R}^+$ is *regularly varying* if there exists $h : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\lim_{t\to\infty} \frac{H(ct)}{H(t)} = h(c)$$

for all $c > 0$. It is well-known that we must then have $h(c) = c^p$ for some $p \in \mathbb{R}$; see, e.g., Theorem 1.1.8 in [24]. Regular variation has played a key role in extreme value theory, where $H = 1 - F$ is then the complement of the cumulative distribution function (*i.e.*, the survival function) of the random variable of interest; see, *e.g.*, [12].

In the context we are considering in the present paper, H will be the convex loss function $\rho \in C$ at hand. Since there is no guarantee that the limit defining regular variation exists[5], we actually need to introduce the map $\gamma : [0,\infty) \to [0,\infty) \cup \{\infty\}$ defined by

$$\gamma(c) := \limsup_{t\to\infty} \frac{\rho(ct)}{\rho(t)}\cdot \tag{4.8}$$

Let $D_\rho := \{c \in [0,\infty) : \gamma(c) < \infty\}$. As we will show, we have a dichotomy: either (a) $D_\rho = [0,\infty)$ or (b) $D_\rho = [0,1]$. In the sequel, we will refer to case (a) and case (b), respectively. We have the following result.

Lemma 4.1. *(i) For all $c \in [0,1]$, $\gamma(c) \le c$, so that $[0,1] \subset D_\rho$. (ii) γ is convex and monotone non-decreasing over D_ρ. (iii) For all $\lambda, c > 0$, $\gamma(\lambda c) \le \gamma(\lambda)\gamma(c)$, so that either case (a) or case (b) holds. (iv) γ is continuous over $[0,\infty)$ in case (a) and over $[0,1)$ in case (b). (v) In case (a), the left-derivative of γ at $c = 1$ exists and satisfies $\gamma'_-(1) \ge 1$.*

Proof. (i) Convexity of ρ, together with the fact that $\rho(0) = 0$, entail that

$$\frac{\rho(ct)}{ct} \le \frac{\rho(t)}{t}$$

for all $c \in (0,1]$ and $t > 0$, so that $\gamma(c) \le c$ for all $c \in (0,1]$. Obviously, $\gamma(0) = 0 \le 0$. (ii) The result follows easily from the fact that ρ is convex and monotone non-decreasing over $[0,\infty)$ (for convexity, one needs to use subadditivity of the lim sup, *i.e.*, $\limsup(f+g) \le \limsup f + \limsup g$). (iii) Fix $\lambda, c > 0$ and let t_k be an arbitrary sequence in $(0,\infty]$ diverging to infinity. Since $\limsup(x_k y_k) \le (\limsup x_k)(\limsup y_k)$ for any non-negative sequences (x_k) and (y_k), we have

$$\gamma(\lambda c) = \limsup_{k\to\infty} \frac{\rho(\lambda c t_k)}{\rho(t_k)} = \limsup_{k\to\infty} \left(\frac{\rho(\lambda c t_k)}{\rho(c t_k)} \frac{\rho(c t_k)}{\rho(t_k)} \right)$$

[5] For the sake of completeness, we will provide in Appendix B an example of a loss function ρ for which this limit indeed does not exist.

$$\leq \left(\limsup_{k\to\infty} \frac{\rho(\lambda c t_k)}{\rho(c t_k)}\right)\left(\limsup_{k\to\infty} \frac{\rho(c t_k)}{\rho(t_k)}\right)$$
$$= \gamma(\lambda)\gamma(c).$$

If $\gamma(c) = \infty$ for all $c > 1$, then Part (i) of the result implies that $D_\rho = [0, 1]$, so that case (b) holds. Otherwise, there exists $c_0 > 1$ such that $\gamma(c_0) < \infty$. Recalling that γ is monotone non-decreasing, the fact that $\gamma(c_0^k) \leq (\gamma(c_0))^k < \infty$ for any positive integer k then entails that $D_\rho = [0, \infty)$, so that case (a) holds. (iv) Convexity of γ over D_ρ entails that γ is continuous over $\mathrm{int}(D_\rho)$, that is, over $(0, \infty)$ in case (a) and over $(0, 1)$ in case (b). Now, since $0 \leq \gamma(c) \leq c$ for any $c \in [0, 1]$, the map γ is also (right-)continuous at zero, which establishes the result. (v) In case (a), γ is convex over $[0, \infty)$, which implies that the left-derivative $\gamma'_-(1)$ exists. It remains to show that $\gamma'_-(1) \geq 1$. *Ad absurdum*, assume that $\gamma'_-(1) < 1$. Then, convexity of γ implies that $\gamma'_-(c) < 1$ for all $c \in (0, 1]$. Since convex and continuous functions are absolutely continuous with (almost everywhere) derivative given by the left- or right-derivative, we have

$$\gamma(1) = \gamma(1) - \gamma(0) = \int_0^1 \gamma'_-(s)\, ds < 1. \tag{4.9}$$

However, the definition of γ trivially entails that $\gamma(1) = 1$, which provides a contradiction. This establishes the result. □

We let

$$p_\rho := \begin{cases} \gamma'_-(1) & \text{in case (a)} \\ \infty & \text{in case (b).} \end{cases} \tag{4.10}$$

Observe that this quantity serves as some analogue of p when $\rho(t) = t^p$ with $p \geq 1$. Indeed, in this case, we have $\gamma(c) = c^p$ for all $c \geq 0$, so that $p_\rho = \gamma'_-(1) = p$. Note, however, that $\rho(t) = t^p \log(1 + t)$, still with $p \geq 1$, provides $\gamma(c) = c^p$ for all $c \geq 0$, too. Obviously, these loss functions correspond to case (a). Consider then $\rho(t) = \exp(t) - 1$, which yields

$$\gamma(c) = \begin{cases} 0 & \text{if } 0 \leq c < 1 \\ 1 & \text{if } c = 1 \\ +\infty & \text{if } c > 1. \end{cases}$$

This corresponds to case (b), for which we let $p_\rho = +\infty$.

Similarly to the definition of γ, we now consider the map $\tilde{\gamma} : [0, 1] \to [0, \infty)$ defined by

$$\tilde{\gamma}(c) = \liminf_{t\to\infty} \frac{\rho(ct)}{\rho(t)}\cdot$$

The properties of this map we will need are summarized in the following result.

Lemma 4.2. *(i) The map $\tilde{\gamma}$ is monotone non-decreasing over $[0, 1]$ and continuous over $[0, 1)$. (ii) In case (a), $\tilde{\gamma}(c) > 0$ for all $c \in (0, 1)$, whereas in case (b), $\tilde{\gamma}(c) = 0$ for all $c \in [0, 1)$.*

Proof. (i) Obviously, Lemma 4.1(i) entails that $\tilde{\gamma}(c) \leq \gamma(c) \leq c$ for all $c \in [0, 1]$, so that $\tilde{\gamma}$ is well-defined. Clearly, $\tilde{\gamma}$ inherits the monotonicity of ρ, too. Let us then show that $\tilde{\gamma}$ is continuous over $[0, 1)$. Trivially, $\tilde{\gamma}(0) = 0$, whereas a direct computation yields that, for all $c \in (0, 1)$,

$$\tilde{\gamma}(c) = \liminf_{t\to\infty} \frac{\rho(ct)}{\rho(t)} = \left(\limsup_{t\to\infty} \frac{\rho(t/c)}{\rho(t)}\right)^{-1} = \begin{cases} (\gamma(1/c))^{-1} & \text{if } \gamma(1/c) \in (0, \infty) \\ 0 & \text{if } \gamma(1/c) = \infty. \end{cases} \tag{4.11}$$

Consider then separately case (a) and case (b)—in each of these, we will show that we cannot have $\gamma(1/c) = 0$ for some $c \in (0, 1)$ in (4.11).

(a) Lemma 4.1(ii) implies that $\gamma(c) \geq \gamma(1) = 1 > 0$ for any $c > 1$, so that $\gamma(c) \in (0, \infty)$ for any $c > 1$ (recall that $\gamma(c) < \infty$ for any $c \geq 0$ in case (a)). Thus, (4.11) yields that $\tilde{\gamma}(c) = 1/\gamma(1/c) > 0$ for any $c \in (0, 1)$. Lemma 4.1(iv) thus implies that $\tilde{\gamma}$ is continuous over $(0, 1)$. Now, in case (a), γ is convex over $[0, \infty)$, with $\gamma'_-(1) \geq 1$, so that $\gamma(c) \to \infty$ as $c \to \infty$. Consequently, (4.11) entails that

$$\lim_{c\to 0} \tilde{\gamma}(c) = \lim_{c\to 0} (\gamma(1/c))^{-1} = 0 = \tilde{\gamma}(0).$$

It follows that $\tilde{\gamma}$ is continuous over $[0, 1)$.

(b) Since $\gamma(c) = \infty$ for all $c > 1$, (4.11) entails that $\tilde{\gamma}(c) = 0$ for all $c \in (0, 1)$. Thus, $\tilde{\gamma}(c) = 0$ for all $c \in [0, 1)$, so that $\tilde{\gamma}$ is continuous over $[0, 1)$.

We have thus proved that, in both cases, $\tilde{\gamma}$ is indeed continuous over $[0, 1)$. Part (ii) of the result was established in our discussion of cases (a)–(b) above. □

Note that, unlike for γ, there is no guarantee that $\tilde{\gamma}$ is convex. The following results will also be useful for our purposes.

Lemma 4.3. *For any nets*[6] $(s_r)_{r>0}$ *and* $(t_r)_{r>0}$ *such that* $s_r/r \to s$ *and* $t_r/r \to t$ *as* $r \to \infty$ *with* $0 \leq s < t < \infty$, *we have*

$$\liminf_{r\to\infty} \frac{\rho(s_r)}{\rho(t_r)} = \tilde{\gamma}\Big(\frac{s}{t}\Big) \qquad \textit{and} \qquad \limsup_{r\to\infty} \frac{\rho(s_r)}{\rho(t_r)} = \gamma\Big(\frac{s}{t}\Big).$$

Proof. For any $\varepsilon \in (0, 1)$, we have

$$\frac{\rho((1-\varepsilon)sr)}{\rho((1+\varepsilon)tr)} \leq \frac{\rho(s_r)}{\rho(t_r)} \leq \frac{\rho((1+\varepsilon)sr)}{\rho((1-\varepsilon)tr)}$$

for all r large enough. An obvious change of variables then provides

$$\tilde{\gamma}\Big(\frac{(1-\varepsilon)s}{(1+\varepsilon)t}\Big) \leq \liminf_{r\to\infty} \frac{\rho(s_r)}{\rho(t_r)} \leq \tilde{\gamma}\Big(\frac{(1+\varepsilon)s}{(1-\varepsilon)t}\Big)$$

and

$$\gamma\Big(\frac{(1-\varepsilon)s}{(1+\varepsilon)t}\Big) \leq \limsup_{r\to\infty} \frac{\rho(s_r)}{\rho(t_r)} \leq \gamma\Big(\frac{(1+\varepsilon)s}{(1-\varepsilon)t}\Big).$$

Since $\tilde{\gamma}$ and γ are continuous maps over $[0, 1)$, letting $\varepsilon \to 0$ establishes the result. □

5. Upper bound

In this section, we derive upper bounds on $\overline{\text{BDP}}(\mu^{\rho}_{\alpha,u})$. Recall the definition of p_ρ in (4.10). Our first main result, that relies on the regular variation properties obtained in the previous section, is the following.

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[6]Recall that a *net* in a topological space X is a function $c\colon (R, \leq) \to X$, where $\leq$ is a pre-order on R; we write $(c_r)_{r\in R}$. It *converges* to $c \in X$ if, for every neighborhood U of c, there exists $r_0 \in R$ such that $r \geq r_0$ implies that $c_r \in U$. Here, $R = \mathbb{R}^+_0$ is equipped with the usual order, and $c_r \to \infty$ is associated with neighborhoods of infinity of the form (a, ∞), $a > 0$.

Theorem 5.1. *Let $\rho \in \mathcal{C}$. Fix $\alpha \in [0,1)$, $u, v \in \mathcal{S}_H$, and $x_1, \ldots, x_n \in H$. If $p_\rho = 1$, then let*

$$\ell := \left\lfloor \frac{n(1 - \alpha \langle u, v \rangle)}{2} \right\rfloor + 1, \tag{5.12}$$

whereas if $p_\rho \in (1, \infty]$, then rather let $\ell = 1$. Then, any M-quantile $\mu^\rho_{\alpha,u}(y_1, \ldots, y_n)$ associated with the contaminated sample defined by

$$y_i := \begin{cases} rv & \text{if } i = 1, \ldots, \ell \\ x_i & \text{if } i = \ell + 1, \ldots, n \end{cases} \tag{5.13}$$

is such that $\|\mu^\rho_{\alpha,u}(y_1, \ldots, y_n)\| \to \infty$ as $r \to \infty$.

This result identifies a number $\ell = \ell_\rho$ of observations it is enough to perturb in direction v to break the M-quantile $\mu^\rho_{\alpha,u}(x_1, \ldots, x_n)$ of order α in direction u. In particular, irrespective of α, u and v, it is enough to perturb only one observation for loss functions such that $p_\rho > 1$. Of course, there is no guarantee at this stage that the value ℓ in (5.12) is optimal when $p_\rho = 1$.

Proof. Fix an arbitrary sequence (r_k) such that $r_k \to \infty$. For any k, pick an arbitrary M-quantile $\mu^\rho_{\alpha,u}(y^k_1, \ldots, y^k_n)$ associated with the contaminated sample in (5.13) with $r = r_k$. We will then show that

$$\|\mu^\rho_{\alpha,u}(y^k_1, \ldots, y^k_n)\| \to \infty \quad \text{as } k \to \infty. \tag{5.14}$$

To this end, it is enough to show that any subsequence of $(\mu^\rho_{\alpha,u}(y^k_1, \ldots, y^k_n))$ admits a further subsequence that diverges to infinity in norm. By abuse of notation, we denote by (r_k) an arbitrary subsequence. At the contaminated sample $y^k_1, \ldots, y^k_n$, the objective function writes

$$M^\rho_{\alpha,u}(\mu; y^k_1, \ldots, y^k_n) = \frac{\ell}{n}\rho(\|r_k v - \mu\|)\Big(1 + \frac{\alpha \langle u, r_k v - \mu\rangle}{\|r_k v - \mu\|}\mathbb{I}[r_k v \neq \mu]\Big)$$
$$+ \frac{1}{n}\sum_{i=\ell+1}^{n} \rho(\|x_i - \mu\|)\Big(1 + \frac{\alpha \langle u, x_i - \mu\rangle}{\|x_i - \mu\|}\mathbb{I}[x_i \neq \mu]\Big).$$

Fix $c \in (0,1)$, to be chosen later. We have

$$M^\rho_{\alpha,u}(cr_k v; y^k_1, \ldots, y^k_n) = \frac{\ell}{n}\rho\big((1-c)r_k\big)(1 + \alpha\langle u, v\rangle)$$
$$+ \frac{1}{n}\sum_{i=\ell+1}^{n} \rho(\|x_i - cr_k v\|)\big(1 - \alpha\langle u, v\rangle\big)(1 + o(1))$$

as $k \to \infty$. Now, for any sequence (μ_k) in $\mathbb{R}^d$ such that $\|\mu_k\| \leq \sqrt{r_k}$ for all k, Lemma 4.3 yields

$$\lim_{k\to\infty} \frac{\rho(\|x_i - \mu_k\|)}{\rho(\|r_k v - \mu_k\|)} \to 0$$

for any $i = 1, \ldots, n$, which allows us to show that

$$M^\rho_{\alpha,u}(\mu_k; y^k_1, \ldots, y^k_n) = \frac{\ell}{n}\rho(\|r_k v - \mu_k\|)(1 + \alpha\langle u, v\rangle)\big(1 + o(1)\big)$$

as $k \to \infty$. For such a sequence (μ_k), we then have

$$\frac{M^{\rho}_{\alpha,u}(cr_k v; y^k_1, \dots, y^k_n)}{M^{\rho}_{\alpha,u}(\mu_k; y^k_1, \dots, y^k_n)} = \frac{\rho\big((1-c)r_k\big)}{\rho(\|r_k v - \mu_k\|)\big(1+o(1)\big)} + \frac{1}{\ell}\sum_{i=\ell+1}^{n} \frac{\rho(\|x_i - cr_k v\|)\big(1-\alpha\,\langle u,v\rangle\big)(1+o(1))}{\rho(\|r_k v - \mu_k\|)(1+\alpha\,\langle u,v\rangle)\big(1+o(1)\big)},$$

so that, letting $M = \max_{i=\ell+1,\dots,n}\|x_i\|$, the monotonicity of ρ yields

$$\frac{M^{\rho}_{\alpha,u}(cr_k v; y^k_1, \dots, y^k_n)}{M^{\rho}_{\alpha,u}(\mu_k; y^k_1, \dots, y^k_n)} \le \frac{\rho\big((1-c)r_k\big)}{\rho(\|r_k v - \mu_k\|)\big(1+o(1)\big)} + \frac{(n-\ell)\rho(M+cr_k)\big(1-\alpha\,\langle u,v\rangle\big)(1+o(1))}{\ell\rho(\|r_k v - \mu_k\|)(1+\alpha\,\langle u,v\rangle)\big(1+o(1)\big)}.$$

Consequently, using the fact that $\liminf(a_k + b_k) \le \limsup(a_k) + \liminf(b_k)$ for any real sequences (a_k) and (b_k), then applying Lemma 4.3, we find

$$\liminf_{k\to\infty} \frac{M^{\rho}_{\alpha,u}(cr_k v; y^k_1, \dots, y^k_n)}{M^{\rho}_{\alpha,u}(\mu_k; y^k_1, \dots, y^k_n)} \le \gamma(1-c) + \frac{(n-\ell)(1-\alpha\,\langle u,v\rangle)}{\ell(1+\alpha\,\langle u,v\rangle)}\tilde{\gamma}(c) =: \theta(c).$$

For now, assume one can choose $c \in (0,1)$ such that $\theta(c) < 1$. Then, there exists $\varepsilon \in (0,1)$ such that, for any sequence (μ_k) with $\|\mu_k\| \le \sqrt{r_k}$ for all k, we have

$$M^{\rho}_{\alpha,u}(cr_k v; y^k_1, \dots, y^k_n) < (1-\varepsilon) M^{\rho}_{\alpha,u}(\mu_k; y^k_1, \dots, y^k_n)$$

for infinitely many k's. Now, choose a sequence (μ_k) of this type such that

$$M^{\rho}_{\alpha,u}(\mu_k; y^k_1, \dots, y^k_n) \le \frac{1}{1-\varepsilon} \inf_{\|\mu\|\le\sqrt{r_k}} M^{\rho}_{\alpha,u}(\mu; y^k_1, \dots, y^k_n).$$

We then have that, still for infinitely many k's,

$$M^{\rho}_{\alpha,u}(cr_k v; y^k_1, \dots, y^k_n) < \inf_{\|\mu\|\le\sqrt{r_k}} M^{\rho}_{\alpha,u}(\mu; y^k_1, \dots, y^k_n).$$

In particular, there exists a subsequence (r_{k_j}) such that, for all j, $\mu^{\rho}_{\alpha,u}(y^{k_j}_1, \dots, y^{k_j}_n)$ belongs to the complement of the ball with radius $\sqrt{r_{k_j}}$, which entails that $(\mu^{\rho}_{\alpha,u}(y^{k_j}_1, \dots, y^{k_j}_n))$ diverges to infinity in norm as $j \to \infty$. Consequently, we showed that any subsequence of $(\mu^{\rho}_{\alpha,u}(y^{k}_1, \dots, y^{k}_n))$ admits a further subsequence that diverges to infinity in norm, which establishes (5.14).

It remains to prove that there indeed exists $c \in (0,1)$ such that $\theta(c) < 1$. Consider first case (b): $p_\rho = \infty$. Then, Lemma 4.2(ii) yields that $\theta(s) = \gamma(1-s)$ for all $s \in (0,1)$, so that Lemma 4.1(i) entails that $\theta(c) < 1$ for any choice $c \in (0,1)$, irrespective of the value of ℓ; in particular, one may take $\ell = 1$ in this case. Turn then to case (a): $p_\rho = \gamma'_-(1) < \infty$. Recall that $\gamma(\lambda s) \le \gamma(\lambda)\gamma(s)$ for all $\lambda, s > 0$ (Lemma 4.1(iii)). Taking $\lambda < 1$, subtracting $\gamma(s)$, dividing by $\lambda s - s$, and taking the limit as $\lambda \nearrow 1$ yields

$$\gamma'_-(s) \ge \gamma'_-(1)\gamma(s)/s = p_\rho \gamma(s)/s \quad \text{for all } s > 0. \tag{5.15}$$

We now proceed with a version of Grönwall's inequality for absolutely continuous functions. Letting $B(s) = p_\rho \log(s)$, $s > 0$, the inequality (5.15) rewrites

$$\gamma'_-(s)e^{-B(s)} \geq B'(s)\gamma(s)e^{-B(s)} \quad \text{for all } s > 0.$$

Since $e^{-B(s)}$ and γ are absolutely continuous on every compact subinterval of $(0, 1)$, their product is as well, which allows us to rewrite the previous inequality as

$$\frac{d}{ds}\big(\gamma(s)e^{-B(s)}\big) \geq 0 \quad \text{for almost every } s > 0.$$

By absolute continuity, $s \mapsto \gamma(s)e^{-B(s)}$ is thus non-decreasing over $[0, \infty)$. In particular,

$$\gamma(s) \leq \gamma(1)e^{B(s)-B(1)} = s^{p_\rho}$$

for all $s \in [0, 1]$. Since $\tilde{\gamma}(s) \leq \gamma(s)$ by definition, it follows that, for all $s \in [0, 1]$,

$$\theta(s) \leq (1-s)^{p_\rho} + \frac{(n-\ell)(1-\alpha\langle u, v\rangle)}{\ell(1+\alpha\langle u, v\rangle)} s^{p_\rho} =: \tilde{\theta}(s).$$

If $p_\rho > 1$, then $\tilde{\theta}'(0) < 0$ so that, because $\tilde{\theta}(0) = 1$, there exists $c \in (0, 1)$ such that $\theta(c) \leq \tilde{\theta}(c) < 1$, irrespective of the value of ℓ; in particular, one may take again $\ell = 1$ in this case. If $p_\rho = 1$, then $\tilde{\theta}'(0) < 0$ as soon as

$$\frac{\ell}{n} > \frac{1-\alpha\langle u, v\rangle}{2}.$$

This concludes the proof. □

Theorem 5.1 provides a complete answer to the breakdown analysis of M-quantiles when $p_\rho \in (1, \infty]$. Indeed, in this case, the result shows that any M-quantile of order α in direction u can be sent to infinity in norm by perturbing a single observation, which yields

$$\big(\underline{\mathrm{BDP}}(\mu^\rho_{\alpha,u}) = \big)\ \overline{\mathrm{BDP}}(\mu^\rho_{\alpha,u}) = \frac{1}{n}.$$

When $p_\rho = 1$, Theorem 5.1 rather provides

$$\overline{\mathrm{BDP}}(\mu^\rho_{\alpha,u}) \leq \left(\left\lfloor \frac{n(1-\alpha)}{2} \right\rfloor + 1\right)/n. \tag{5.16}$$

In this case, however, there might still be some margin left for improvement. In the important case $\rho(t) = t$ (which provides $p_\rho = 1$), it was established in Corollary 2.2 of [20] that, when $n(1-\alpha)/2$ is not an integer, (5.16) is an equality. We will show in the next section (see Theorem 6.1) that this extends to loss functions ρ that are asymptotically linear in the sense that the quantity

$$a_\rho := \lim_{t\to\infty} \frac{\rho(t)}{t} \in (0, \infty) \cup \{\infty\} \tag{5.17}$$

is actually finite.[7] As shown by the loss function defined by $\rho(t) = t\log(1+t)$, however, it may be the case that $p_\rho = 1$ and $a_\rho = \infty$, which leaves a gap in our analysis of the breakdown. This motivates the following result, that, in the case where $p_\rho = 1$ and $a_\rho = \infty$, provides an upper bound on $\overline{\mathrm{BDP}}(\mu^\rho_{\alpha,u})$ that improves over (5.16).

[7] Note that convexity of ρ entails that the function $t \mapsto \rho(t)/t$ is non-decreasing, which guarantees existence of the limit in (5.17). This limit is obviously positive since $\rho(t)/t > 0$ for all $t > 0$.

Theorem 5.2. *Let $\rho \in \mathcal{C}$ be such that the quantity a_ρ in (5.17) is infinite. Fix $\alpha \in [0, 1)$, $u \in \mathcal{S}_H$, and $x_1, \ldots, x_n \in H$. Then, any M-quantile $\mu^\rho_{\alpha,u}(y_1, \ldots, y_n)$ associated with the contaminated sample defined by*

$$y_i := \begin{cases} ru & \text{if } i = 1 \\ x_i & \text{if } i = 2, \ldots, n \end{cases}$$

is such that $\|\mu^\rho_{\alpha,u}(y_1, \ldots, y_n)\| \to \infty$ as $r \to \infty$.

Proof. Let $(c_r)_{r>0}$ be a net, to be determined later, such that $c_r \le r/3$ and $c_r \to \infty$ as $r \to \infty$. We first show that

$$\inf_{\|\mu\| \le c_r} M^\rho_{\alpha,u}(\mu; y_1, \ldots, y_n) \ge \frac{1+\alpha}{n}\rho(r - c_r). \tag{5.18}$$

Trivially, for any $\|\mu\| \le c_r (< r)$,

$$M^\rho_{\alpha,u}(\mu; y_1, \ldots, y_n) \ge \frac{1}{n}\rho(\|ru - \mu\|)\Big(1 + \frac{\alpha\,\langle u, ru - \mu\rangle}{\|ru - \mu\|}\Big). \tag{5.19}$$

Decomposing μ into $\mu = \langle u, \mu\rangle\, u + \{\mu - \langle u, \mu\rangle\, u\}$, we have

$$\|ru - \mu\| = \sqrt{(r - \langle u, \mu\rangle)^2 + \|\mu\|^2 - \langle u, \mu\rangle^2} = \sqrt{r^2 - 2r\,\langle u, \mu\rangle + \|\mu\|^2}.$$

For any fixed value of $\langle u, \mu\rangle$ in $[-c_r, c_r]$, the function

$$\mu \mapsto \frac{1}{n}\rho(\|ru - \mu\|)\Big(1 + \frac{\alpha\,\langle u, ru - \mu\rangle}{\|ru - \mu\|}\Big) = \frac{1}{n}\rho(\|ru - \mu\|) + \frac{\alpha(r - \langle u, \mu\rangle)\rho(\|ru - \mu\|)}{n\|ru - \mu\|} \tag{5.20}$$

is minimized when $\|ru - \mu\|$, or equivalently $\|\mu\|$, is minimized (since convexity entails that $\rho(r)/r$ is monotone non-decreasing), which is the case when $\mu = \langle u, \mu\rangle\, u$. The minimum of the function in (5.20) over $\|\mu\| \le c_r$ is thus

$$\min_{s \in [-c_r, c_r]} \frac{1+\alpha}{n}\rho(r - s) = \frac{1+\alpha}{n}\rho(r - c_r)$$

Jointly with (5.19), this proves (5.18).

Now, we have

$$M^\rho_{\alpha,u}(2c_r u; y_1, \ldots, y_n) = \frac{1+\alpha}{n}\rho(r - 2c_r) + R_n,$$

where we wrote

$$R_n := \frac{1}{n}\sum_{i=2}^{n} \rho(\|x_i - 2c_r u\|)\Big(1 + \frac{\alpha\,\langle u, x_i - 2c_r u\rangle}{\|x_i - 2c_r u\|}\mathbb{I}[x_i \ne 2c_r u]\Big).$$

Note that, for $M = \max_{i=2,\ldots,n} \|x_i\|$, we have

$$R_n \le \frac{(1+\alpha)(n-1)}{n}\rho(2c_r + M),$$

so that

$$\Big(\inf_{\|\mu\| \le c_r} M^\rho_{\alpha,u}(\mu; y_1, \ldots, y_n)\Big) - M^\rho_{\alpha,u}(2c_r u; y_1, \ldots, y_n)$$

$$\geq \frac{1+\alpha}{n}(\rho(r-c_r)-\rho(r-2c_r)) - \frac{(1+\alpha)(n-1)}{n}\rho(2c_r+M)$$
$$= \frac{(1+\alpha)\rho(2c_r+M)}{n}\left(\frac{\rho(r-c_r)-\rho(r-2c_r)}{\rho(2c_r+M)} - (n-1)\right).$$

Note that if

$$\frac{\rho(r-c_r)-\rho(r-2c_r)}{\rho(2c_r+M)} \to \infty \ \text{ as } r \to \infty, \tag{5.21}$$

then, for r large enough, any M-quantile $\mu^{\rho}_{\alpha,u}(y_1,\dots,y_n)$ will be outside the ball $\|\mu\| \leq c_r$, which will establish the result since $c_r \to \infty$ as $r \to \infty$. Thus, it only remains to prove (5.21).

Since ρ is absolutely continuous with (almost everywhere) non-decreasing and non-negative left-derivative ψ_-, we have, for r large enough such that $c_r \geq M$,

$$\frac{\rho(r-c_r)-\rho(r-2c_r)}{\rho(2c_r+M)} \geq \frac{\rho(r-c_r)-\rho(r-2c_r)}{\rho(3c_r)} \geq \frac{\psi_-(r-2c_r)c_r}{\rho(3c_r)}.$$

The convexity of ρ implies that, for all $t > 0$,

$$\varphi(t) := \frac{\rho(t)}{t} = \frac{\rho(t)-\rho(0)}{t} \leq \psi_-(t).$$

It follows that

$$\frac{\rho(r-c_r)-\rho(r-2c_r)}{\rho(2c_r+M)} \geq \frac{\varphi(r-2c_r)c_r}{\rho(3c_r)} = \frac{\varphi(r-2c_r)c_r}{\varphi(3c_r)3c_r} = \frac{\varphi(r-2c_r)}{3\varphi(3c_r)}.$$

Let $f : [0,\infty) \to [0,\infty)$ be a continuous and strictly monotone increasing majorant of ρ with $f(0) = 0$ (in particular, f is a continuous one-to-one map and $f(t) \to \infty$ as $t \to \infty$), and assume, for now, that one can choose $c_r \to \infty$ in such a way that $c_r \leq r/3$ and $f(3c_r) \leq \varphi(r-2c_r)$ for all $r > 0$. Then,

$$\frac{\varphi(r-2c_r)}{\varphi(3c_r)} \geq \frac{f(3c_r)}{\varphi(3c_r)} \geq \frac{\rho(3c_r)}{\varphi(3c_r)} = 3c_r.$$

Since $c_r \to \infty$, we deduce that

$$\frac{\rho(r-c_r)-\rho(r-2c_r)}{\rho(2c_r+M)} \to \infty$$

as $r \to \infty$. It remains to show that there exists such a net $(c_r)_{r>0}$. For this purpose, let

$$c_r := \sup A_r, \qquad \text{with } \ A_r := \left\{s \in [0, r/3] : 3s \leq f^{-1}(\varphi(r-2s))\right\}.$$

For all $r > 0$, we have $0 \in A_r$, so that A_r is non-empty and c_r is well-defined. Trivially, $c_r \leq r/3$ for all $r > 0$. Observe then that $c_r \leq c_{r'}$ whenever $r \leq r'$. Indeed, notice first that since ρ is continuous over $(0,\infty)$, the function $t \mapsto \varphi(t) = \rho(t)/t$ is continuous over $(0,\infty)$, too. In addition, the convexity of ρ entails that φ is non-decreasing over $(0,\infty)$. Consequently, for all $r \leq r'$, we have

$$3c_r \leq f^{-1}(\varphi(r-2c_r)) \leq f^{-1}(\varphi(r'-2c_r)).$$

It follows that, for all $r \leq r'$, we have $c_r \in A_{r'}$, hence $c_r \leq c_{r'}$. Assume now, *ad absurdum*, that c_r is bounded; $c_r \leq B$ for all $r > 0$, say. For any $r > 3B$, we would then have

$$3s > f^{-1}(\varphi(r-2s)),$$

for all $s \in (B, r/3]$. Since $\varphi(t) \to \infty$ as $t \to \infty$ by assumption, taking then $r \to \infty$ with s fixed in the last display yields a contradiction. We deduce that $c_r \to \infty$ as $r \to \infty$, which concludes the proof. □

Despite its generality in terms of loss functions ρ, Theorem 5.2 restricts to contamination in the direction u in which M-quantiles are computed. This is in contrast with Theorem 5.1 that provides a directional analysis of the breakdown: recall that this result revealed in particular that, when $p_\rho > 1$, quantiles break as soon as a single observation is perturbed *in an arbitrary direction*. Also, when $p_\rho = 1$, Theorem 5.1 interestingly provides an upper bound on the BDP that depends on the contamination direction v.

More importantly, Theorems 5.1–5.2 together lead to the following result.

Corollary 5.1. *Let $\rho \in \mathcal{C}$. Fix $\alpha \in [0, 1)$, $u \in \mathcal{S}_H$, and $x_1, \dots, x_n \in H$. Then,*

$$\overline{\mathrm{BDP}}(\mu^\rho_{\alpha,u}; x_1, \dots, x_n) \le \left(\left\lfloor \frac{n(1-\alpha)}{2} \right\rfloor + 1 \right) \frac{1}{n} \tag{5.22}$$

if a_ρ is finite, whereas

$$\underline{\mathrm{BDP}}(\mu^\rho_{\alpha,u}; x_1, \dots, x_n) = \overline{\mathrm{BDP}}(\mu^\rho_{\alpha,u}; x_1, \dots, x_n) = \frac{1}{n}$$

if a_ρ is infinite.

Proof. If a_ρ is finite, then we trivially obtain that $\gamma(c) = c$ for all $c \ge 0$. Therefore, $p_\rho = 1$, so that the conclusion follows from Theorem 5.1. If $a_\rho = \infty$, then the conclusion directly follows from Theorem 5.2. □

This result completely characterizes the breakdown point of M-quantiles when a_ρ is infinite and it shows that the breakdown point is then $1/n$. In particular, this is the case for the loss function that served as the motivation for deriving Theorem 5.2, namely the loss function defined by $\rho(t) = t \log(1 + t)$. When a_ρ is finite, however, a lower bound is needed, which is the content of the next section.

6. Lower bound

In the important particular case considered in (5.22) for which the loss function ρ is eventually linear, the results of the previous section only provide an upper bound on the BDP, rather than the BDP itself. An important example is the Huber loss function defined by $\rho_c(t) = (t^2/2)\mathbb{I}[0 < t < c] + c(t - (c/2))\mathbb{I}[t \ge c]$, with $c > 0$. In this section, we obtain an essentially matching lower bound. We have the following result.

Theorem 6.1. *Let $\rho \in \mathcal{C}$ be such that the quantity a_ρ in (5.17) is finite. Fix $\alpha \in [0, 1)$ and $u \in \mathcal{S}_H$. Then,*

$$\underline{\mathrm{BDP}}(\mu^\rho_{\alpha,u}; x_1, \dots, x_n) \ge \left\lceil \frac{n(1-\alpha)}{2} \right\rceil / n$$

for any sample $x_1, \dots, x_n \in H$.

Proof. We start with the following preliminary remarks. Since $t \mapsto \rho(t)/t$ is non-decreasing, (5.17) ensures that $\rho(t) \leq a_\rho t$ for all $t \geq 0$. The function g defined by $g(t) := a_\rho t - \rho(t)$ for all $t \geq 0$ is thus non-negative, concave, satisfies $g(0) = 0$ and $g(t)/t \to 0$ as $t \to \infty$. Also, $t \mapsto g(t)/t$ is non-increasing over $(0, \infty)$. For any $t_0 > 0$, concavity of g entails that

$$t \mapsto \frac{g(t) - g(t_0)}{t - t_0}$$

is monotone non-increasing over $[0, \infty) \setminus \{t_0\}$, so, denoting as $g'_-(t_0)$ (resp., $g'_+(t_0)$) the left-derivative (resp., right-derivative) of g at t_0,

$$\frac{g(s) - g(t_0)}{s - t_0} \geq g'_-(t_0) \geq g'_+(t_0) \geq \frac{g(t) - g(t_0)}{t - t_0} \tag{6.23}$$

for any $s \in [0, t_0)$ and $t \in (t_0, \infty)$. This implies that

$$g(t) \leq g(t_0) + g'_-(t_0)(t - t_0) \tag{6.24}$$

for any $t \geq 0$. Dividing by $t (> 0)$ and letting $t \to \infty$ thus gives $g'_-(t_0) \geq 0$. Since t_0 is arbitrary, $g'_-(t) \geq 0$ for all $t > 0$, so that Lemma A.2 entails that g is non-decreasing.

Let us then proceed with the proof. Fix arbitrary $x_1, \ldots, x_n \in H$. Since $\mu^\rho_{\alpha,u}$ is trivially translation-equivariant, Lemma 2.1 from [21] implies that

$$\mathrm{BDP}(\mu^\rho_{\alpha,u}; x_1 + z, \ldots, x_n + z) = \mathrm{BDP}(\mu^\rho_{\alpha,u}; x_1, \ldots, x_n)$$

for any $z \in H$. Therefore, we may assume that $\mu^\rho_{\alpha,u}(x_1, \ldots, x_n) = 0$. Now, fix a positive integer

$$\ell \leq \left\lceil \frac{n(1-\alpha)}{2} \right\rceil - 1.$$

Then, $\ell/n < (1-\alpha)/2$, so that we can pick $\varepsilon > 0$ small enough to have

$$\frac{\ell}{n} < \frac{1-\alpha}{2} - \varepsilon(1 + 2\alpha). \tag{6.25}$$

Fix an arbitrary sample $y_1, \ldots, y_n \in H$ differing from $x_1, \ldots, x_n$ by at most ℓ observations. To keep the notation light, we denote the corresponding quantiles by $\mu^{\rho,x}_{\alpha,u}$ and $\mu^{\rho,y}_{\alpha,u}$. Since $\mu^\rho_{\alpha,u}$ is invariant under permutations of its arguments, we may assume that $y_i = x_i$ for all $i = \ell + 1, \ldots, n$. Letting $P_n = n^{-1} \sum_{i=1}^n \delta_{y_i}$ denote the empirical probability measure associated with the sample $y_1, \ldots, y_n$, then define the function

$$\begin{aligned} \mu \mapsto \Delta^\rho_{\alpha,u}(\mu; P_n) &:= M^\rho_{\alpha,u}(\mu; P_n) - M^\rho_{\alpha,u}(0; P_n) \\ &= \frac{1}{n} \sum_{i=1}^n \left\{ \rho(\|y_i - \mu\|)\left(1 + \alpha \frac{\langle u, y_i - \mu\rangle}{\|y_i - \mu\|} \mathbb{I}[y_i \neq \mu]\right) - \rho(\|y_i\|)\left(1 + \alpha \frac{\langle u, y_i\rangle}{\|y_i\|} \mathbb{I}[y_i \neq 0]\right) \right\}. \end{aligned}$$

We prove the result by showing that there exists a positive real number R, that does not depend on the contaminated sample $y_1, \ldots, y_n$, such that $\Delta^\rho_{\alpha,u}(\mu; P_n) > 0$ as soon as $\|\mu\| > R$.

Since $g(t)/t \to 0$ as $t \to \infty$,

$$T_\varepsilon := \inf\left\{t > 0 : g(s) \leq a_\rho \varepsilon s \text{ for all } s \geq t\right\} \tag{6.26}$$

is well-defined (recall that $\varepsilon > 0$ was fixed in (6.25)). Note that since $t \mapsto g(t)/t$ is non-increasing on $(0, \infty)$, we also have

$$T_\varepsilon = \inf\left\{t > 0 : g(t) \le a_\rho \varepsilon t\right\} = \inf\left\{t > 0 : \rho(t)/t \ge a_\rho(1-\varepsilon)\right\}. \tag{6.27}$$

Now, the identity $\rho(t) = a_\rho t - g(t)$ provides, for any $\mu \in H$,

$$\begin{aligned}
\Delta^\rho_{\alpha,u}(\mu; P_n) = \frac{1}{n}\sum_{i=1}^n a_\rho\big(\|y_i-\mu\| - \|y_i\| - \alpha\,\langle u,\mu\rangle\,\big) \\
+\frac{1}{n}\sum_{i=1}^n \Bigg\{g(\|y_i\|)\bigg(1+\alpha\frac{\langle u,y_i\rangle}{\|y_i\|}\mathbb{I}[y_i\neq 0]\bigg) \\
-g(\|y_i-\mu\|)\bigg(1+\alpha\frac{\langle u,y_i-\mu\rangle)}{\|y_i-\mu\|}\mathbb{I}[y_i\neq\mu]\bigg)\Bigg\}\mathbb{I}[\|y_i\|\le T_\varepsilon] \\
+\frac{1}{n}\sum_{i=1}^n \Bigg\{g(\|y_i\|)\bigg(1+\alpha\frac{\langle u,y_i\rangle}{\|y_i\|}\mathbb{I}[y_i\neq 0]\bigg) \\
-g(\|y_i-\mu\|)\bigg(1+\alpha\frac{\langle u,y_i-\mu\rangle}{\|y_i-\mu\|}\mathbb{I}[y_i\neq\mu]\bigg)\Bigg\}\mathbb{I}[\|y_i\|> T_\varepsilon] \\
=: \Delta_1 + \Delta_2^- + \Delta_2^+,
\end{aligned}$$

say. Letting $M = \max_{i=\ell+1,\dots,n}\|x_i\| = \max_{i=\ell+1,\dots,n}\|y_i\|$, the triangle inequality yields

$$\big|\|y_i-\mu\| - \|y_i\| - \alpha\,\langle u,\mu\rangle\,\big| \le (1+\alpha)\|\mu\|$$

for $i = 1,\dots,\ell$, and

$$\|y_i-\mu\| - \|y_i\| - \alpha\,\langle u,\mu\rangle \ge \|\mu\| - 2M - \alpha\|\mu\|,$$

for $i = \ell+1,\dots,n$. Since $a_\rho > 0$, we thus have

$$\begin{aligned}
\Delta_1 &\ge a_\rho\bigg\{-\frac{\ell}{n}(1+\alpha)\|\mu\| + \frac{n-\ell}{n}\Big((1-\alpha)\|\mu\| - 2M\Big)\bigg\} \\
&= a_\rho\|\mu\|\Big(1-\alpha-\frac{2\ell}{n}\Big) - \frac{2a_\rho M(n-\ell)}{n}.
\end{aligned} \tag{6.28}$$

Let us now turn to Δ_2^- and fix $\mu \in H$ with $\|\mu\| > T_\varepsilon$. Since g is concave and $g(0) \ge 0$, it is well-known that g is subadditive, that is $g(t_1+t_2) \le g(t_1) + g(t_2)$ for all $t_1, t_2 \ge 0$ (for the sake of completeness, we prove this in Lemma A.4). In particular, since $g(0) = 0$ and g is non-decreasing, we have

$$\begin{aligned}
\Delta_2^- &\ge -\frac{1}{n}\sum_{i=1}^n g(\|y_i-\mu\|)(1+\alpha)\mathbb{I}[\|y_i\|\le T_\varepsilon] \\
&\ge -(1+\alpha)g(T_\varepsilon + \|\mu\|)\frac{1}{n}\sum_{i=1}^n \mathbb{I}[\|y_i\|\le T_\varepsilon] \\
&\ge -(1+\alpha)\big(g(T_\varepsilon) + g(\|\mu\|)\big)
\end{aligned}$$

$$\geq -(1+\alpha)g(T_\varepsilon) - a_\rho\varepsilon(1+\alpha)\|\mu\|, \tag{6.29}$$

where the last inequality follows from the fact that $\|\mu\| > T_\varepsilon$. Finally, Δ_2^+ rewrites

$$\Delta_2^+ = \frac{1}{n}\sum_{i=1}^n \left\{ g(\|y_i\|)\Big(1+\alpha\frac{\langle u, y_i\rangle}{\|y_i\|}\Big) - g(\|y_i-\mu\|)\Big(1+\alpha\frac{\langle u, y_i-\mu\rangle}{\|y_i-\mu\|}\mathbb{I}[y_i\neq\mu]\Big)\right\}\mathbb{I}[\|y_i\| > T_\varepsilon]$$

(since $\|y_i\| > T_\varepsilon$ implies that $y_i \neq 0$). From (6.23) with $s=0$, we have

$$\frac{g(\|y_i\|)}{\|y_i\|} \geq g'_-(\|y_i\|)$$

so that (6.24) implies that

$$\begin{aligned} g(\|y_i-\mu\|) &\leq g(\|y_i\|) + g'(\|y_i\|)\big(\|y_i-\mu\| - \|y_i\|\big) \\ &\leq g(\|y_i\|) + \frac{g(\|y_i\|)}{\|y_i\|}\|\mu\|. \end{aligned} \tag{6.30}$$

Using (6.30), we then have

$$\begin{aligned} \Delta_2^+ &\geq \frac{1}{n}\sum_{i=1}^n \left\{ g(\|y_i\|)\alpha\left\langle u, \frac{y_i}{\|y_i\|} - \frac{y_i-\mu}{\|y_i-\mu\|}\mathbb{I}[y_i\neq\mu]\right\rangle\Big) - \frac{g(\|y_i\|)}{\|y_i\|}(1+\alpha)\|\mu\|\right\}\mathbb{I}[\|y_i\| > T_\varepsilon] \\ &\geq -\frac{1}{n}\sum_{i=1}^n \left\{ g(\|y_i\|)\alpha\frac{2\|\mu\|}{\|y_i\|} + \frac{g(\|y_i\|)}{\|y_i\|}(1+\alpha)\|\mu\|\right\}\mathbb{I}[\|y_i\| > T_\varepsilon] \\ &\geq -a_\rho\varepsilon(1+3\alpha)\|\mu\|, \end{aligned} \tag{6.31}$$

where we used the fact that

$$\left\| \frac{v}{\|v\|} - \frac{w}{\|w\|}\mathbb{I}[w\neq 0]\right\| \leq \frac{2\|v-w\|}{\|v\|}$$

for any $v \in H\setminus\{0\}$ (this directly follows from Lemma A.1). Combining (6.28), (6.29) and (6.31) shows that, for all $\mu \in H$ such that $\|\mu\| > T_\varepsilon$,

$$\begin{aligned} \Delta^\rho_{\alpha,u}(\mu; P_n) &\geq a_\rho\|\mu\|\Big(1-\alpha-\frac{2\ell}{n}\Big) - \frac{2a_\rho M(n-\ell)}{n} - (1+\alpha)g(T_\varepsilon) - 2a_\rho\varepsilon(1+2\alpha)\|\mu\| \\ &= A_\varepsilon\|\mu\| - B_\varepsilon, \end{aligned}$$

with

$$A_\varepsilon := 2a_\rho\left(\frac{1-\alpha}{2} - \frac{\ell}{n} - \varepsilon(1+2\alpha)\right) > 0 \quad\text{and}\quad B_\varepsilon := \frac{2a_\rho M(n-\ell)}{n} + (1+\alpha)g(T_\varepsilon)$$

(positivity of A_ε follows from (6.25)). Thus, $\Delta^\rho_{\alpha,u}(\mu; P_n) = M^\rho_{\alpha,u}(\mu, P_n) - M^\rho_{\alpha,u}(0, P_n) > 0$ as soon as

$$\|\mu\| > \max(T_\varepsilon, B_\varepsilon/A_\varepsilon) =: R_\varepsilon.$$

It follows that any M-quantile $\mu^{\rho,y}_{\alpha,u}$ associated with the contaminated sample $y_1,\dots,y_n$ satisfies $\|\mu^{\rho,y}_{\alpha,u}\| \leq R_\varepsilon$. Since R_ε is independent of the contamination (it depends only on the uncontaminated sample $x_1,\dots,x_n$), we have

$$\sup_y \|\mu^{\rho,x}_{\alpha,u} - \mu^{\rho,y}_{\alpha,u}\| = \sup_y \|\mu^{\rho,y}_{\alpha,u}\| \leq R_\varepsilon < \infty.$$

This implies that $\underline{\text{BDP}}(\mu^\rho_{\alpha,u}; x_1,\dots,x_n) \geq (\ell+1)/n$, which concludes the proof. □

The proof actually provides a quantitative estimate for any $\ell \in \{1, 2, \ldots, \lceil \frac{n(1-\alpha)}{2} \rceil - 1\}$. Continuity of g implies that $g(T_\varepsilon) = a_\rho \varepsilon T_\varepsilon$, so that we established the bound

$$\sup_y \|\mu^{\rho,x}_{\alpha,u} - \mu^{\rho,y}_{\alpha,u}\| \leq \inf \left\{ R_\varepsilon : 0 < \varepsilon < \varepsilon_{\sup} := \tfrac{1-\alpha}{2(1+2\alpha)} - \tfrac{\ell}{n(1+2\alpha)} \right\} \tag{6.32}$$

(the upper-bound for $\varepsilon > 0$ originates from (6.25)), where

$$R_\varepsilon = \max \left(T_\varepsilon, \frac{M(1 - \frac{\ell}{n}) + \frac{\varepsilon(1+\alpha)T_\varepsilon}{2}}{\frac{1-\alpha}{2} - \frac{\ell}{n} - \varepsilon(1 + 2\alpha)} \right)$$

involves $T_\varepsilon = \inf\{t > 0 : \rho(t)/t \geq a_\rho(1 - \varepsilon)\}$; see (6.27). For $\rho(t) = t$, we have $T_\varepsilon = 0$ for any $\varepsilon > 0$, so that the bound in (6.32) is

$$\sup_y \|\mu^{\rho,x}_{\alpha,u} - \mu^{\rho,y}_{\alpha,u}\| \leq \frac{M(1 - \frac{\ell}{n})}{\frac{1-\alpha}{2} - \frac{\ell}{n}} = M + \frac{M(\frac{1+\alpha}{2})}{\frac{1-\alpha}{2} - \frac{\ell}{n}}. \tag{6.33}$$

Consider then the Huber loss function $\rho_c(t) = (t^2/2)\mathbb{I}[0 < t < c] + c(t - (c/2))\mathbb{I}[t \geq c]$, with $c > 0$. Note that $\varepsilon_{\sup} < 1/2$, so that $T_\varepsilon = c/(2\varepsilon)$, which yields

$$R_\varepsilon = \max \left(\frac{c}{2\varepsilon}, \frac{M(1 - \frac{\ell}{n}) + \frac{c(1+\alpha)}{4}}{\frac{1-\alpha}{2} - \frac{\ell}{n} - \varepsilon(1 + 2\alpha)} \right).$$

An easy calculation then provides

$$\sup_y \|\mu^{\rho_c,x}_{\alpha,u} - \mu^{\rho_c,y}_{\alpha,u}\| \leq M + \frac{M(\frac{1+\alpha}{2}) + \frac{c(3+5\alpha)}{4}}{\frac{1-\alpha}{2} - \frac{\ell}{n}}.$$

Note that, as expected, letting $c \to 0$ provides the bound in (6.33), whereas, letting $c \to \infty$ makes the bound infinite even for $\ell = 1$, which is in line with the fact for $\rho(t) = t^2$, the BDP is $1/n$ (Corollary 5.1).

More importantly, Corollary 5.1 and Theorem 6.1 readily yield the following result.

Corollary 6.1. *Let $\rho \in \mathcal{C}$. Fix $\alpha \in [0, 1)$, $u \in \mathcal{S}_H$, and $x_1, \ldots, x_n \in H$. Then,*

$$\underline{\mathrm{BDP}}(\mu^\rho_{\alpha,u}; x_1, \ldots, x_n) = \overline{\mathrm{BDP}}(\mu^\rho_{\alpha,u}; x_1, \ldots, x_n) = \left\lceil \frac{n(1-\alpha)}{2} \right\rceil \Big/ n$$

if a_ρ is finite and $n(1-\alpha)/2$ is not an integer,

$$\left\lceil \frac{n(1-\alpha)}{2} \right\rceil \Big/ n \leq \underline{\mathrm{BDP}}(\mu^\rho_{\alpha,u}; x_1, \ldots, x_n) \leq \overline{\mathrm{BDP}}(\mu^\rho_{\alpha,u}; x_1, \ldots, x_n) \leq \left(\left\lceil \frac{n(1-\alpha)}{2} \right\rceil + 1 \right) \Big/ n$$

if a_ρ is finite and $n(1-\alpha)/2$ is an integer, whereas

$$\underline{\mathrm{BDP}}(\mu^\rho_{\alpha,u}; x_1, \ldots, x_n) = \overline{\mathrm{BDP}}(\mu^\rho_{\alpha,u}; x_1, \ldots, x_n) = \frac{1}{n}$$

if a_ρ is infinite.

A direct consequence of this result is that, for loss functions that are eventually linear (still in the sense that $a_\rho = 1$), the asymptotic breakdown point of M-quantiles coincides with the breakdown point of spatial quantiles, namely $(1 - \alpha)/2$. This applies, in particular, to the M-quantiles resulting from the Huber loss functions.

7. Final comments

In this work, we established the existence of M-quantiles in (possibly infinite-dimensional) Hilbert spaces under extremely mild conditions. We also obtained essentially complete results on the breakdown point of these M-quantiles. For loss functions that are not eventually linear, the breakdown point is $1/n$, whereas for loss functions that are eventually linear, it is $\lceil n(1-\alpha)/2\rceil/n$ provided that $n(1-\alpha)/2$ is not an integer. It is thus only for finitely many values of α, and in the eventually linear case, that the finite-sample breakdown point was not fully determined. While this in principle leaves room for further improvement, some comments are in order. First, when the breakdown point was not obtained exactly, the positive difference between our upper bound for the upper breakdown point and our lower bound for the lower breakdown point takes the minimal possible value, namely $1/n$. In particular, the asymptotic breakdown point is equal to $(1-\alpha)/2$ in that case, too. Second, a close inspection of the results from [20] reveals that, even for the standard spatial quantiles obtained with $\rho(t) = t$, the breakdown point when $n(1-\alpha)/2$ is not an integer actually depends on the particular representative one needs to choose when uniqueness fails; that is, depending on the choice of this representative, the breakdown point may be $\lceil n(1-\alpha)/2\rceil/n$ or $(\lceil n(1-\alpha)/2\rceil+1)/n$. This was fixed in [20] by deriving breakdown point results for a given representative, namely the barycenter of the collection of spatial quantiles, for which the breakdown point was shown there to be $\lceil n(1-\alpha)/2\rceil/n$. However, it is important to note that this approach cannot be adopted in the framework of M-quantiles: indeed, as mentioned in Section 2, the collection of M-quantiles may be non-convex, so that the barycenter of the collection of M-quantiles may fail to be an M-quantile itself. In this sense, the results obtained in this paper, that are in any case essentially complete, leave even less space for improvement than it may seem at first.

Appendix A: Auxiliary results

In this first appendix, we state and prove some results that were used in the proofs of Lemma 2.1 and Theorem 6.1. We start by stating the following two preliminary results.

Lemma A.1 (Lemma S.1.2 in [19]). *Let $v, w \in \mathbb{R}^d \setminus \{0\}$. Then*

$$\left\| \frac{v}{\|v\|} - \frac{w}{\|w\|} \right\| \leq 2 \min\left(\frac{\|v-w\|}{\|v\|}, \frac{\|v-w\|}{\|w\|} \right).$$

Lemma A.2 ([23]). *Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Assume that f is left-differentiable on (a, b), with left-derivative f'_-. Then,*

$$f'_-(c_-) \leq \frac{f(b) - f(a)}{b - a} \leq f'_-(c_+)$$

for some $c_-, c_+ \in (a, b)$.

These results allow us to establish the following lemma, that was used in the proof of Lemma 2.1.

Lemma A.3. *Let $\rho \in \mathcal{C}$ and $P \in \mathcal{P}_H^\rho$. Fix $\alpha \in [0, 1)$ and $u \in \mathcal{S}_H$. Let (μ_k) be a sequence in H that converges weakly to some $\mu \in H$. Then, there exists a P-integrable function $g : H \mapsto \mathbb{R}^+$ such that*

$$H^\rho_{\alpha,u}(z - \mu_k) - H^\rho_{\alpha,u}(z) \geq -g(z)$$

for all $z \in H$ *and all* k.

Proof. Write

$$
\begin{aligned}
H^{\rho}_{\alpha,u}(z-\mu_k) - H^{\rho}_{\alpha,u}(z)
&= \rho(\|z-\mu_k\|)\Big(1+\alpha\frac{\langle u, z-\mu_k\rangle}{\|z-\mu_k\|}\xi_{z,\mu_k}\Big) - \rho(\|z\|)\Big(1+\alpha\frac{\langle u, z\rangle}{\|z\|}\xi_{z,0}\Big) \\
&= (\rho(\|z-\mu_k\|)-\rho(\|z\|))\Big(1+\alpha\frac{\langle u, z-\mu_k\rangle}{\|z-\mu_k\|}\xi_{z,\mu_k}\Big) \\
&\quad +\alpha\rho(\|z\|)\Big(\frac{\langle u, z-\mu_k\rangle}{\|z-\mu_k\|}\xi_{z,\mu_k} - \frac{\langle u, z\rangle}{\|z\|}\Big)\xi_{z,0}.
\end{aligned}
$$

where the last equality results from the fact that $\rho(0)=0$. So,

$$
H^{\rho}_{\alpha,u}(z-\mu_k) - H^{\rho}_{\alpha,u}(z) = T_{k1}(z)+T_{k2}(z)+T_{k3}(z),
$$

with

$$
T_{k1}(z) := (\rho(\|z-\mu_k\|)-\rho(\|z\|))\Big(1+\alpha\frac{\langle u, z-\mu_k\rangle}{\|z-\mu_k\|}\xi_{z,\mu_k}\Big),
$$

$$
T_{k2}(z) := \alpha\rho(\|z\|)\frac{\langle u, z\rangle}{\|z\|}(\xi_{z,\mu_k}-1)\xi_{z,0} \geq -\alpha\rho(\|\mu_k\|)\mathbb{I}[z=\mu_k],
$$

and

$$
T_{k3}(z) := \alpha\rho(\|z\|)\Big(\frac{\langle u, z-\mu_k\rangle}{\|z-\mu_k\|} - \frac{\langle u, z\rangle}{\|z\|}\Big)\xi_{z,\mu_k}\xi_{z,0}.
$$

Since $\mu_k \rightharpoonup \mu$, the uniform boundedness principle entails that $(\|\mu_k\|)$ is a bounded sequence. With $C = \sup\|\mu_k\|$, the fact that ρ is monotone non-decreasing then entails that

$$
T_{k2}(z) \geq -\alpha\rho(C),
$$

where the lower bound is trivially P-integrable. Now, Lemma A.1 and the fact that ψ_- is monotone non-decreasing (as the left-derivative of a convex function), we obtain, with δ_0 as in (2.4),

$$
\begin{aligned}
T_{k3}(z) &\geq -2\alpha\|\mu_k\|\{\rho(\|z\|)/\|z\|\}\xi_{z,0} \\
&\geq -2\alpha C\psi_-(\|z\|) \\
&\geq -2\alpha C\psi_-(\|z\|+\delta_0),
\end{aligned}
$$

where the lower bound is P-integrable since $P \in \mathcal{P}^{\rho}_H$. Finally, we turn to T_{k1}, whose treatment is more complicated. If $\|z-\mu_k\| > \|z\|$, then Lemma A.2 ensures that there exists $c_k^- \in (\|z\|, \|z-\mu_k\|)$ such that

$$
\rho(\|z-\mu_k\|)-\rho(\|z\|) \geq \psi_-(c_k^-)\big(\|z-\mu_k\|-\|z\|\big) \geq \psi_-(\|z\|)\big(\|z-\mu_k\|-\|z\|\big),
$$

whereas if $\|z-\mu_k\| < \|z\|$, then the same result yields that there exists $c_k^+ \in (\|z-\mu_k\|, \|z\|)$ for which

$$
\rho(\|z\|)-\rho(\|z-\mu_k\|) \leq \psi_-(c_k^+)\big(\|z\|-\|z-\mu_k\|\big) \leq \psi_-(\|z\|)\big(\|z\|-\|z-\mu_k\|\big).
$$

Thus, we have

$$\rho(\|z-\mu_k\|)-\rho(\|z\|)\geq\psi_-(\|z\|)\big(\|z-\mu_k\|-\|z\|\big)$$

for all $z\in H$, which yields

$$T_{k1}(z)\geq\psi_-(\|z\|)\big(\|z-\mu_k\|-\|z\|\big)\left(1+\alpha\frac{\langle u,z-\mu_k\rangle}{\|z-\mu_k\|}\right)\xi_{z,\mu_k}.$$

The triangle inequality and the fact that $\|\mu_k\|\leq C$ thus provide

$$\begin{aligned}T_{k1}(z) &\geq -(1+\alpha)C\psi_-(\|z\|)\\ &\geq -(1+\alpha)C\psi_-(\|z+\delta_0\|),\end{aligned}$$

where the lower bound is P-integrable since $P\in\mathcal{P}_H^\rho$. The result is proved. □

Finally, the following result was used in the proof of Theorem 6.1.

Lemma A.4. *Let $g:\mathbb{R}^+\to\mathbb{R}$ be concave and satisfy $g(0)\geq 0$. Then, $g(t_1+t_2)\leq g(t_1)+g(t_2)$ for all $t_1,t_2\geq 0$.*

Proof. For $t_1=0$ or $t_2=0$, the result is trivial. Fix then $t_1,t_2>0$. Concavity of g yields

$$\frac{t_1}{t_1+t_2}g(0)+\frac{t_2}{t_1+t_2}g(t_1+t_2)\leq g(t_2),$$

which, using $g(0)\geq 0$, provides

$$\frac{t_2}{t_1+t_2}g(t_1+t_2)\leq g(t_2). \tag{A.34}$$

Since exchanging t_1 and t_2 yields

$$\frac{t_1}{t_1+t_2}g(t_1+t_2)\leq g(t_1), \tag{A.35}$$

the result follows by adding up (A.34)–(A.35). □

Appendix B: Need for non-standard regular variation theory

In this second appendix, we provide an example that demonstrates that it was needed to use the concepts of lim sup and lim inf in the results of Section 4, so that standard regular variation theory does not apply to the general setting considered in the present work. Let us focus on maps ρ defined by

$$\rho(t):=\begin{cases}\exp(f(\log t)) & \text{if } t>0\\ 0 & \text{if } t=0,\end{cases}$$

where $f:\mathbb{R}\to\mathbb{R}$ satisfies $\lim_{s\to-\infty}f(s)=-\infty$ (this ensures continuity of ρ over $[0,\infty)$). The power loss function $\rho(t)=t^p$ is obtained for $f(s)=ps$, and the exponential loss $\rho(t)=\exp(t)-1$ corresponds to $f(s)=\log(\exp(\exp(s))-1)$. Let us examine sufficient conditions to ensure that ρ is convex. Further assuming that f is twice differentiable, ρ is twice differentiable over $(0,\infty)$, with

$$\rho''(t)=\frac{\rho(t)}{t^2}\Big\{f''(\log t)+\big(f'(\log t)\big)^2-f'(\log t)\Big\}$$

for all $t > 0$. Consequently, ρ is convex if and only if

$$\kappa_f(s) := f''(s) + (f'(s))^2 - f'(s) \geq 0 \tag{B.36}$$

for all $s \in \mathbb{R}$. For the power loss function $\rho(t) = t^p$, corresponding to $f(s) = ps$, we have

$$\kappa_f(s) = p^2 - p \geq 0$$

for all $s \in \mathbb{R}$ if and only if $p \geq 1$; we further have $\inf_{s\in\mathbb{R}} \kappa_p(s) > 0$ as soon as $p > 1$. In particular, when f_0 satisfies the positivity condition (B.36) strictly, one expects that small perturbations of f_0 yield a convex loss function ρ. This is the content of the next result.

Proposition B.1. *Let $f_0 : \mathbb{R} \to \mathbb{R}$ be such that $\lim_{s\to-\infty} f_0(s) = -\infty$. Assume that f_0 is twice differentiable and that*

$$\kappa_0 := \inf_{s\in\mathbb{R}} \kappa_{f_0}(s) > 0.$$

Then, for any twice differentiable map $g : \mathbb{R} \to \mathbb{R}$ with bounded first and second derivatives such that $\limsup_{s\to-\infty} g(s) < \infty$, there exists $\delta_0 = \delta_0(f_0, g) > 0$ such that, for all $|\delta| < \delta_0$, the map defined by

$$\rho_\delta(t) = \begin{cases} \exp(f_\delta(\log t)) & \text{if } t > 0 \\ 0 & \text{if } t = 0, \end{cases} \qquad \text{with } f_\delta = f_0 + \delta g,$$

is convex over $[0, \infty)$.

Proof. The fact that $\lim_{s\to-\infty} f_0(s) = -\infty$ and $\limsup_{s\to-\infty} g(s) < \infty$ entails that

$$\lim_{s\to-\infty} f_\delta(s) = \lim_{s\to-\infty} (f_0(s) + \delta g(s)) = -\infty,$$

so that ρ_δ is continuous over $[0, \infty)$. From (B.36), we know that ρ_δ is convex if and only if $\kappa_{f_\delta}(s) \geq 0$ for all $s \in \mathbb{R}$. A direct computation provides

$$\kappa_{f_\delta}(s) = \Big(f_0''(s) + (f_0'(s))^2 - f_0'(s)\Big) + \delta\Big((2f_0'(s) - 1)g'(s) + g''(s)\Big) + \delta^2(g'(s))^2.$$

Consequently, letting $\|\varphi\|_\infty := \sup_{s\in\mathbb{R}} |\varphi(s)|$ for any map $\varphi : \mathbb{R} \to \mathbb{R}$, we have

$$\kappa_{f_\delta}(s) \geq \kappa_0 - |\delta|\Big((2\|f_0'\|_\infty + 1)\|g'\|_\infty + \|g''\|_\infty\Big) =: \kappa_0 - C(f_0, g)|\delta| \geq 0,$$

as soon as $|\delta| \leq \kappa_0 / C(f_0, g)$, which concludes the proof. □

Inspection of the proof of Proposition B.1 shows that, when $f_0'(s) \geq 1/2$ (which is the case for $f_0(s) = ps$ and $p \geq 1$ arising from considering $\rho(t) = t^p$), the result holds provided g' and g'' are only lower-bounded by some real constant; in particular, g' and g'' can take arbitrarily large positive values. For $f(s) = ps + \delta g(s)$, Proposition B.1 entails that one obtains the family of convex loss functions $\{\rho_{p,g}\}$, indexed by $p > 1$ and $g : \mathbb{R} \to \mathbb{R}$ twice differentiable with $g' \geq -B$ and $g'' \geq -B$ for some $B > 0$, such that $\limsup_{s\to-\infty} g(s) < \infty$, defined by

$$\rho_{p,g}(t) := t^p \exp(\delta g(\log t)),$$

for all $t > 0$ and $|\delta|$ small enough. For such loss functions, we have for all $c > 0$ and $t > 0$

$$\frac{\rho(ct)}{\rho(t)} = c^p \exp\big(\delta g(\log(ct)) - \delta g(\log t)\big).$$

Since we are interested in the behavior of this ratio when $t \to \infty$ and $c > 0$ is fixed, we may consider the reparametrization $t(u) = e^u$, $u > 0$, which leads to studying the behavior of

$$\frac{\rho(ct(u))}{\rho(t(u))} = c^p \exp\big(\delta g(u + \log c) - \delta g(u)\big).$$

Heuristically, this ratio will display a non-monotone behavior when $g(s)$ oscillates (*i.e.*, the derivative of g changes sign) and keeps doing so as $s \to \infty$. Consider then $g(s) = \sin(2\beta s)/2$ with $\beta > 0$, and notice that g satisfies the assumptions of Proposition B.1. Using the identity $\sin(a) - \sin(b) = 2\cos((a+b)/2)\sin((a-b)/2)$, we obtain

$$\frac{\rho(ct(u))}{\rho(t(u))} = c^p \exp\big(\delta \sin(\beta \log c)\cos(2\beta u + \beta \log c)\big).$$

Further reparametrizing by $v = 2\beta u + \beta \log c$ yields

$$\frac{\rho(ct(v))}{\rho(t(v))} = c^p \exp\big(\delta \sin(\beta \log c)\cos(v)\big).$$

Thus, for all $c > 0$, we have

$$\gamma(c) = \limsup_{v\to\infty} \frac{\rho(ct(v))}{\rho(t(v))} = c^p \exp\big(|\delta \sin(\beta \log c)|\big)$$

and

$$\tilde{\gamma}(c) = \liminf_{v\to\infty} \frac{\rho(ct(v))}{\rho(t(v))} = c^p \exp\big(-|\delta \sin(\beta \log c)|\big).$$

This confirms that relying on the classical theory of regular variation would have reduced the generality of our findings, which provides a motivation for the results derived in Section 4.

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