Discussion of "Invariant Co-ordinate Selection" by David E. Tyler, Frank Critchley, Lutz Dümbgen, and Hannu Oja

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Beyond the role it plays in detecting departures from ellipticity, ICS is potentially useful to choose a proper model for the data at hand among the many multivariate models available in the literature: (mixtures of) elliptical models, the independent component (IC) models of Section 5.2 (see also Nordhausen et al. 2009), skew-elliptical models (see, e.g., Genton 2004), etc. This discussion partly supports this claim by proposing an informal graphical method that allows to "test" the null hypothesis $\mathcal{H}_0^{\mathrm{IC}}$ under which IC models are appropriate.

In Fig1(c), it is shown how a couple of location-scatter estimates $(\hat{\mu}_\ell, \hat{V}_\ell)$, $\ell = 1, 2$ can be used to detect departures from ellipticity, based on the fact that, for any such couple and under ellipticity, one should have $d_i(\hat{\mu}_2, \hat{V}_2) \approx \lambda\, d_i(\hat{\mu}_1, \hat{V}_1)$ for some $\lambda > 0$. For $\mathcal{H}_0^{\mathrm{IC}}$, one could similarly think of using three—or four—different scatter estimates to derive—typically, via Theorem 5—a couple of consistent estimates $\hat{H}_\ell$, $\ell = 1, 2$ for the underlying mixing matrix H (clearly, it is crucial to adopt a common normalization for $\hat{H}_1$, $\hat{H}_2$, and H here, such as, e.g., the Z-standardization in the R-package ICS; see Nordhausen et al. 2008 for details). Although proper (Frobenius-type) distances between the resulting $\hat{H}_1$ and $\hat{H}_2$ would provide natural test statistics for $\mathcal{H}_0^{\mathrm{IC}}$, a direct graphical tool, in the same spirit as in Fig1(c), is the scatter plot of *ICS distances* $(d_i^{\mathrm{ICS}}(\hat{H}_1), d_i^{\mathrm{ICS}}(\hat{H}_2))$, $i = 1, \ldots, n$, with

$$d_i^{\mathrm{ICS}}(\hat{H}_\ell) := \sqrt{(\hat{H}_\ell' Y_i - \hat{\mu}_\ell^{\mathrm{ICS}})'(\hat{\Lambda}_\ell^{\mathrm{ICS}})^{-2}(\hat{H}_\ell' Y_i - \hat{\mu}_\ell^{\mathrm{ICS}})},$$

where $\hat{\mu}_\ell^{\mathrm{ICS}}$ is the vector of marginal medians for the ℓth ICS and $\hat{\Lambda}_\ell^{\mathrm{ICS}}$ is the diagonal matrix collecting the corresponding marginal median absolute deviations. Under $\mathcal{H}_0^{\mathrm{IC}}$, all points in such scatter plots should roughly sit on the main diagonal, which allows to detect possible violations of $\mathcal{H}_0^{\mathrm{IC}}$.

The choice of the various scatter matrices is, here as well, a delicate issue. But one might still argue that combining scatter matrices with different robustness properties could reveal interesting features. This is illustrated (with the same data as in Section 6.1) in the four figures below, where it should be noted that, interestingly, only the plot based exclusively on robust scatter matrices seems to be compatible with $\mathcal{H}_0^{\mathrm{IC}}$.

As shown beautifully in the paper, though, the relevance of ICS extends far beyond IC models, and I would like to congratulate the authors for one of the most refreshing and inspiring works of the decade in the field of Multivariate Statistics.

References

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[2] Nordhausen, K., Oja, H. and Paindaveine, D. (2009) Signed-rank tests for location in the symmetric independent component model. *Journal of Multivariate Analysis* **100**, 821–834.

[3] Nordhausen, K., Oja, H. and Tyler, D. E. (2008) Tools for exploring multivariate data via ICS/ICA. In R Package, Version 1.1–1. (Available from http://cran.r-project.org.)