On Multivariate Runs Tests for Randomness

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Abstract

This paper proposes several extensions of the concept of *runs* to the multivariate setup, and studies the resulting tests of multivariate randomness against serial dependence. Two types of multivariate runs are defined: (i) an elliptical extension of the spherical runs proposed by Marden (1999), and (ii) an original concept of matrix-valued runs. The resulting runs tests themselves exist in various versions, one of which is a function of the number of data-based hyperplanes separating pairs of observations only. All proposed multivariate runs tests are affine-invariant and highly robust: in particular, they allow for heteroskedasticity and do not require any moment assumption. Their limiting distributions are derived under the null hypothesis and under sequences of local vector ARMA alternatives. Asymptotic relative efficiencies with respect to Gaussian Portmanteau tests are computed, and show that, while Marden-type runs tests suffer severe consistency problems, tests based on matrix-valued

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runs perform uniformly well for moderate-to-large dimensions. A Monte-Carlo study confirms the theoretical results and investigates the robustness properties of the proposed procedures. A real data example is also treated, and shows that combining both types of runs tests may provide some insight on the reason why rejection occurs, hence that Marden-type runs tests, despite their lack of consistency, also are of interest for practical purposes.

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1 INTRODUCTION.

Runs tests for randomness are among the oldest nonparametric procedures. Some of them are based on the length of the longest run, while others—as in Wald and Wolfowitz (1940)—are based on the number of runs. The latter can typically be described as follows. Fix some $\theta \in \mathbb{R}$ and denote by $\mathcal{H}_\theta^{(n)}$ the hypothesis under which the observations $X_1, \ldots, X_n$ are mutually independent random variables with $\mathrm{P}[X_t < \theta] = \mathrm{P}[X_t \leq \theta] = \frac{1}{2}$, $t = 1, \ldots, n$. Writing then $U_{t,\theta} := I_{[X_t > \theta]} - I_{[X_t < \theta]}$ for the sign of the "residual" $X_t - \theta$, where I_A stands for the indicator function of set A, the number of runs in the sequence $X_1, \ldots, X_n$ is defined as $R_\theta^{(n)} := 1 + \sum_{t=2}^n I_{[U_{t,\theta} \neq U_{t-1,\theta}]}$, that is, with probability one under $\mathcal{H}_\theta^{(n)}$, as the number of blocks of consecutive 1's or -1's in the sequence $U_{1,\theta}, \ldots, U_{n,\theta}$. Since a large (resp., small) value of $R_\theta^{(n)}$ clearly indicates negative (resp., positive) serial dependence, classical Wald-Wolfowitz-type runs tests reject the null of randomness when $|R_\theta^{(n)} - \mathrm{E}_0[R_\theta^{(n)}]|$ is too large, where E_0 denotes the expectation under the null. Noting that $R_\theta^{(n)} = 1 + \sum_{t=2}^n (1 - U_{t,\theta} U_{t-1,\theta})/2$ almost surely under $\mathcal{H}_\theta^{(n)}$, we have that $\mathrm{E}_0[R_\theta^{(n)}] = (n+1)/2$ and that (still under $\mathcal{H}_\theta^{(n)}$)

$$r_\theta^{(n)} := \frac{-2(R_\theta^{(n)} - \mathrm{E}_0[R_\theta^{(n)}])}{\sqrt{n-1}} = \frac{1}{\sqrt{n-1}} \sum_{t=2}^n U_{t,\theta} U_{t-1,\theta} \tag{1.1}$$

is asymptotically standard normal. The resulting runs test therefore rejects the null of randomness, at asymptotic level α, as soon as $Q_\theta^{(n)} := (r_\theta^{(n)})^2 > \chi^2_{1,1-\alpha}$ (throughout, $\chi^2_{\ell,1-\alpha}$ denotes the upper-α quantile of the χ^2_ℓ distribution).

This explains how runs tests of randomness can be performed when the common median θ of the observations is specified. Now, if (as often) θ is unknown, one rejects the null of randomness for large values of $(r_{\hat\theta}^{(n)})^2$, where $\hat\theta$ is an adequate estimator of θ. Note that under $\mathcal{H}_\theta^{(n)}$, it is trivial to derive the exact distribution of $(U_{1,\theta}, \ldots, U_{n,\theta})$, so that even exact runs tests can be defined in the θ-specified case. If θ is not known, only the asymptotic version above can be implemented, though.

Of course, runs tests are highly robust procedures. They are clearly insensitive to outlying values in the series $|X_1 - \theta|, \ldots, |X_n - \theta|$, and can deal with *heteroskedasticity* (and even with different classes of marginal distributions, provided that the common median is θ). This is in sharp contrast with most tests for randomness, including classical Portmanteau tests based on standard autocorrelations, which require stationary processes. A disadvantage of runs tests with respect to such Portmanteau tests, however, is that they can detect serial dependence at lag one only. To improve on that, Dufour et al. (1998) introduced *generalized runs tests* which reject the null of randomness $\mathcal{H}_\theta^{(n)}$ at asymptotic level α whenever

$$Q_{H,\theta}^{(n)} := \sum_{h=1}^{H} (r_{h,\theta}^{(n)})^2 := \sum_{h=1}^{H} \left(\frac{1}{\sqrt{n-h}} \sum_{t=h+1}^{n} U_{t,\theta} U_{t-h,\theta} \right)^2 > \chi^2_{H,1-\alpha}, \tag{1.2}$$

where H is a fixed positive integer. Such tests, parallel to standard Portmanteau tests, can clearly detect serial dependence up to lag H. Finally, performances of runs tests are very high under heavy tails, yet can be poor under medium and light tails.

To sum up, runs tests provide very robust and easily implementable nonparametric tests of randomness for *univariate* series. Now, in a world where multivariate time series and panel data belong to the daily practice of time series analysis, it is natural to try to define runs tests of *multivariate* randomness, which is the objective of this paper. Parallel to the multivariate extension of the companion concepts of signs and ranks, the extension of runs to the multivariate setup is, however, a very delicate issue. If one forgets about the multivariate runs and runs tests introduced in Friedman and Rafsky (1979), which relate to two-sample location problems, the only concept of multivariate runs in the literature—to the best of the author's knowledge—was proposed by Marden (1999), in an excellent paper that not only reviews multivariate nonparametric rank-based techniques, but also introduces some original ideas, including these new multivariate runs.

Considering a series of k-variate observations $\mathbf{X}_1, \ldots, \mathbf{X}_n$, Marden's multivariate runs generalize (1.1) into

$$r_{\boldsymbol{\theta}}^{(n)} := \frac{1}{\sqrt{n-1}} \sum_{t=2}^{n} \mathbf{U}'_{t,\boldsymbol{\theta}} \mathbf{U}_{t-1,\boldsymbol{\theta}}, \tag{1.3}$$

where $\mathbf{U}_{t,\boldsymbol{\theta}} := (\mathbf{X}_t - \boldsymbol{\theta})/\|\mathbf{X}_t - \boldsymbol{\theta}\|$ is the so-called *spatial sign* of $\mathbf{X}_t$ with respect to $\boldsymbol{\theta}$; see, e.g., Möttönen and Oja (1995). As Marden (1999) points out, these runs have a clear intuitive interpretation as measures of multivariate serial correlation, since the magnitude of $r_{\boldsymbol{\theta}}^{(n)}$ will tend to be large if either consecutive observations tend to be pointing in the same direction or in opposite directions (with respect to $\boldsymbol{\theta}$ in both cases). Mimicking the classical univariate runs procedure, the resulting test, $\phi^{(n)}_{\text{Marden}}$ say, then rejects the null of randomness for large values of $(r_{\boldsymbol{\theta}}^{(n)})^2$; see Section 3.

Clearly, $\phi^{(n)}_{\text{Marden}}$ nicely extends classical runs tests to the multivariate setup, but is not *affine-invariant*, which is a property that often plays an important role in multivariate nonparametric statistics. Also, most properties of $\phi^{(n)}_{\text{Marden}}$ remain unclear, since only the asymptotic null distribution was derived in Marden (1999). This calls, in particular, for a detailed derivation of the asymptotic properties of this test (including a careful investigation of its performances with respect to daily-practice multivariate Portmanteau tests), as well as for a study of its finite-sample properties. All this was of course beyond the scope of the Marden (1999) review paper.

The present paper defines affine-invariant versions of the Marden runs in (1.3), and studies the asymptotic and finite-sample properties of the resulting multivariate runs tests. Affine-invariance is achieved either by substituting standardized spatial signs for the spatial signs in (1.3) (see, e.g., Taskinen et al. (2003, 2005), Larocque et al. (2007), and Hallin and Paindaveine (2008)) or by using the hyperplane-based multivariate signs known as *interdirections* (see Randles 1989). When based on the latter, the proposed multivariate runs tests only use the fact that pairs of data points are separated (or not) by data-based hyperplanes of $\mathbb{R}^k$. To further improve on $\phi^{(n)}_{\mathrm{Marden}}$, which can detect lag one serial dependence only, we also define *generalized multivariate runs tests* in the spirit of Dufour et al. (1998). Throughout, we consider both the $\boldsymbol{\theta}$-specified and $\boldsymbol{\theta}$-unspecified cases. As we will see, the Marden-type runs tests suffer a severe lack of power against a broad class of alternatives. This motivates the introduction of alternative multivariate runs of the form

$$\mathbf{r}^{(n)\mathrm{f}}_{\boldsymbol{\theta}} := \frac{1}{\sqrt{n-1}} \sum_{t=2}^{n} \mathbf{U}_{t,\boldsymbol{\theta}} \mathbf{U}'_{t-1,\boldsymbol{\theta}}, \tag{1.4}$$

which are obtained by substituting *outer products* of spatial signs for inner products in (1.3), a substitution that is of course superfluous in the univariate case. For dimension $k \geq 2$, however, the resulting runs tests, which reject the null of randomness when some appropriate norm of the (matrix-valued) runs $\mathbf{r}^{(n)\mathrm{f}}_{\boldsymbol{\theta}}$ is too large, do not suffer the same lack of power as Marden-type runs tests, and actually enjoy very good asymptotic properties. In particular, we will establish that the infimum of their asymptotic relative efficiencies (AREs) with respect to classical Portmanteau tests is given by $(\frac{k-1}{k})^2$ in dimension k. This clearly shows that the main drawback of univariate runs tests, namely their possible lack of efficiency, vanishes as k increases.

The outline of the paper is as follows. Section 2 describes the null hypothesis of multivariate randomness we are considering. Two families of multivariate runs are defined in Section 3: *shape runs*, which are based on standardized spatial signs

(Section 3.1) and *hyperplane-based runs*, which make use of Randles' interdirections (Section 3.2). For each family, affine-invariant Marden-type runs tests and full-rank runs tests are defined in the $\boldsymbol{\theta}$-specified case. Section 3.3 then discusses the extension to the $\boldsymbol{\theta}$-unspecified case. A real data example is treated in Section 4. The asymptotic properties of the proposed tests are investigated in Section 5, both under the null (Section 5.1) and under local vector ARMA alternatives (Section 5.2), while Section 5.3 derives the AREs of our tests with respect to their standard Portmanteau competitors. A Monte-Carlo study is presented in Section 6. Eventually, some final comments are given in Section 7, and an appendix collects technical proofs.

2 MULTIVARIATE RANDOMNESS.

The null hypothesis we consider in this paper is a multivariate extension of the one described in the Introduction. More precisely, for any k-vector $\boldsymbol{\theta}$ and any matrix $\mathbf{V}$ in the collection $\mathcal{M}_k$ of positive definite symmetric $k \times k$ matrices with trace k, denote by $\mathcal{H}^{(n)}_{\boldsymbol{\theta},\mathbf{V}}$ the hypothesis under which the k-variate observations $\mathbf{X}_1, \ldots, \mathbf{X}_n$ do satisfy

$$(\mathbf{X}_1', \ldots, \mathbf{X}_n')' \stackrel{\mathcal{D}}{=} ((\boldsymbol{\theta} + d_1 \mathbf{V}^{1/2}\mathbf{U}_1)', \ldots, (\boldsymbol{\theta} + d_n \mathbf{V}^{1/2}\mathbf{U}_n)')'$$

(throughout, $\mathbf{V}^{1/2}$ stands for the positive definite symmetric root of $\mathbf{V}$ and $\stackrel{\mathcal{D}}{=}$ denotes equality in distribution), where the $\mathbf{U}_t$'s are i.i.d. random vectors that are uniformly distributed over the unit sphere $\mathcal{S}^{k-1} \subset \mathbb{R}^k$, and the d_t's are arbitrary positive random variables. As announced above, we will consider the null of randomness with a specified center and with an unspecified center, namely $\mathcal{H}^{(n)}_{\boldsymbol{\theta}} := \bigcup_{\mathbf{V}\in\mathcal{M}_k} \mathcal{H}^{(n)}_{\boldsymbol{\theta},\mathbf{V}}$ and $\mathcal{H}^{(n)} := \bigcup_{\boldsymbol{\theta}\in\mathbb{R}^k} \bigcup_{\mathbf{V}\in\mathcal{M}_k} \mathcal{H}^{(n)}_{\boldsymbol{\theta},\mathbf{V}}$, respectively.

Although $\mathcal{H}^{(n)}_{\boldsymbol{\theta}}$ and $\mathcal{H}^{(n)}$ will be regarded as hypotheses of multivariate *randomness*, the observations $\mathbf{X}_t$ may actually be stochastically dependent under the null, since the d_t's—unlike the $\mathbf{U}_t$'s—are not assumed to be mutually independent. It might therefore be more appropriate to speak about *sign (or directional) randomness*. However, we will simply speak of *randomness* in the sequel, since this terminology is standard and broadly accepted in the field. Clearly, combining a test of (univariate) randomness on the d_t's with the multivariate runs tests proposed in this paper, might be appropriate if one wants to test the null hypothesis of "full" randomness.

Note further that the observations need not be identically distributed, and that the distribution of $\mathbf{X}_t$ is elliptically symmetric only if d_t and $\mathbf{U}_t$ are independent; if the latter fail being independent, it is sometimes said that $\mathbf{X}_t$ has a distribution *with elliptical directions*, which includes certain skewed distributions (see Randles 1989, 2000).

The so-called *shape matrix* $\mathbf{V}$ is such that, under $\mathcal{H}^{(n)}_{\boldsymbol{\theta},\mathbf{V}}$, the unit k-vectors

$$\mathbf{U}_{t,\boldsymbol{\theta}}(\mathbf{V}) := \frac{\mathbf{V}^{-1/2}(\mathbf{X}_t - \boldsymbol{\theta})}{\|\mathbf{V}^{-1/2}(\mathbf{X}_t - \boldsymbol{\theta})\|}, \quad t = 1, \ldots, n, \tag{2.1}$$

are i.i.d. uniformly distributed over $\mathcal{S}^{k-1}$. If $\mathbf{X}_t$ has an elliptical distribution, the level sets of its characteristic function $\mathbf{z} \mapsto \mathrm{E}[\exp(i\mathbf{z}'\mathbf{X}_t)]$ (and, in the absolutely continuous case, those of its pdf; see below) are the hyperellipsoids $\{(\mathbf{z}-\boldsymbol{\theta})'\mathbf{V}^{-1}(\mathbf{z}-\boldsymbol{\theta}) = c : \mathbf{z} \in \mathbb{R}^k\}$, $c > 0$. The *shape* and orientation of these ellipsoids are determined by $\mathbf{V}$, which justifies the terminology. Still, the shape matrix $\mathbf{V}$ need not be (a multiple of) the covariance matrix of (any of) the observations; since the definitions above do not require any moment assumption, covariance matrices indeed may not exist.

Due to the important role it plays in the local alternatives we consider in Section 5.2, the case where the $\mathbf{X}_t$'s are i.i.d. with an absolutely continuous (with respect to the Lebesgue measure on $\mathbb{R}^k$) elliptically symmetric distribution is of particular interest. The common pdf of the observations is then of the form

$$\mathbf{x} \mapsto (\omega_k \mu_{k-1,f})^{-1} (\det \mathbf{V})^{-1/2} f\left(\sqrt{(\mathbf{x}-\boldsymbol{\theta})'\mathbf{V}^{-1}(\mathbf{x}-\boldsymbol{\theta})}\right), \tag{2.2}$$

where $\boldsymbol{\theta} \in \mathbb{R}^k$ is the location center, $\mathbf{V} \in \mathcal{M}_k$ is the shape matrix, ω_k stands for the $(k-1)$-dimensional Lebesgue measure of the unit sphere $\mathcal{S}^{k-1}$, and where, letting $\mu_{\ell,f} := \int_0^\infty r^\ell f(r)\, dr$, the so-called *radial density function* $f : \mathbb{R}_0^+ \to \mathbb{R}^+$ satisfies $\mu_{k-1,f} < \infty$. The Mahalanobis distance $d_{t,\boldsymbol{\theta}}(\mathbf{V}) := \|\mathbf{V}^{-1/2}(\mathbf{X}_t - \boldsymbol{\theta})\|$ between $\mathbf{X}_t$ and $\boldsymbol{\theta}$ (in the metric associated with $\mathbf{V}$) admits the pdf $r \mapsto (\mu_{k-1,f})^{-1} r^{k-1} f(r)\, I_{[r>0]}$. Finally, the density in (2.2) admits finite second-order moments iff $\mu_{k+1,f} < \infty$.

Particular k-variate elliptical distributions of interest below are the multinormal distributions, with radial densities $f(r) := \sigma^{-1} \exp(-r^2/2\sigma^2)$, the Student distributions, with radial densities $f(r) := \sigma^{-1}(1 + \nu^{-1}(r/\sigma)^2)^{-(k+\nu)/2}$ (for $\nu > 0$ degrees of freedom), and the power-exponential distributions, with radial densities $f(r) := \sigma^{-1} \exp(-(r/\sigma)^{2\eta})$, $\eta > 0$; in each case, $\sigma > 0$ is an arbitrary scale factor.

3 AFFINE-INVARIANT RUNS TESTS.

As mentioned in the Introduction, Marden (1999) proposes a multivariate runs test for randomness, which, using the notation introduced in the previous section, is based on runs of the form

$$\sum_{t=2}^{n} \mathbf{U}'_{t,\boldsymbol{\theta}}(\mathbf{I}_k)\mathbf{U}_{t-1,\boldsymbol{\theta}}(\mathbf{I}_k), \tag{3.1}$$

where $\mathbf{I}_k$ stands for the k-dimensional identity matrix. More precisely, the resulting test $\phi^{(n)}_{\text{Marden}}$ rejects the null of randomness $\mathcal{H}^{(n)}_{\boldsymbol{\theta}}$ at asymptotic level α as soon as $(ns^{(n)})^{-1}(\sum_{t=2}^{n} \mathbf{U}'_{t,\boldsymbol{\theta}}(\mathbf{I}_k)\mathbf{U}_{t-1,\boldsymbol{\theta}}(\mathbf{I}_k))^2 > \chi^2_{1,1-\alpha}$, where $s^{(n)} := \text{tr}[((n-1)^{-1} \sum_{t=1}^{n} \mathbf{U}_{t,\boldsymbol{\theta}}(\mathbf{I}_k)\, \mathbf{U}'_{t,\boldsymbol{\theta}}(\mathbf{I}_k))^2]$. This test is valid (in the sense that it does meet the asymptotic level constraint) even if the elliptical direction assumption of the previous section is weakened into the assumption that the $\mathbf{U}_{t,\boldsymbol{\theta}}(\mathbf{I}_k)$'s are i.i.d. with a common centrally symmetric distribution ($-\mathbf{U}_{t,\boldsymbol{\theta}}(\mathbf{I}_k)$ and $\mathbf{U}_{t,\boldsymbol{\theta}}(\mathbf{I}_k)$ share the same distribution). Marden (1999) also considers the $\boldsymbol{\theta}$-unspecified case, in which he proposes replacing $\boldsymbol{\theta}$ with the *spatial median* $\hat{\boldsymbol{\theta}}_{\text{spat}}$ of the $\mathbf{X}_t$'s, that is, with the solution of the M-estimating equation $\frac{1}{n} \sum_{t=1}^{n} \mathbf{U}_{t,\hat{\boldsymbol{\theta}}_{\text{spat}}}(\mathbf{I}_k) = \mathbf{0}$.

However, as all statistical procedures based on the so-called *spatial signs* $\mathbf{U}_{t,\boldsymbol{\theta}}(\mathbf{I}_k)$ of the observations with respect to $\boldsymbol{\theta}$ (see, e.g., Möttönen and Oja 1995 or Nevalainen et al. 2007), $\phi^{(n)}_{\text{Marden}}$ is *not* affine-invariant, but only orthogonal invariant. This implies that the relative performances of $\phi^{(n)}_{\text{Marden}}$ with respect to standard Gaussian Portmanteau tests crucially depend on the underlying shape matrix $\mathbf{V}$, and, even more importantly, that the decision to reject or not the null of randomness may be affected by a change of units in any univariate series. To improve on this, we propose substituting the *elliptical* Marden runs

$$\sum_{t=2}^{n} \mathbf{U}'_{t,\boldsymbol{\theta}}(\mathbf{V})\mathbf{U}_{t-1,\boldsymbol{\theta}}(\mathbf{V}) \tag{3.2}$$

for the original Marden runs in (3.1) (which are clearly of a *spherical* nature). As announced earlier, we also plan to define alternative multivariate runs, obtained by replacing *inner products* with *outer products* in the runs above; such runs, in their elliptical versions, are thus of the form

$$\sum_{t=2}^{n} \mathbf{U}_{t,\boldsymbol{\theta}}(\mathbf{V})\mathbf{U}'_{t-1,\boldsymbol{\theta}}(\mathbf{V}). \tag{3.3}$$

While we refer to the runs in (3.2) as *Marden runs* (since their orthogonal-invariant version was first proposed in Marden 1999), the matrix-valued runs in (3.3) throughout will be called *full-rank runs* (this terminology will become clear later on).

Under $\mathcal{H}^{(n)}_{\boldsymbol{\theta}}$, the shape matrix $\mathbf{V}$ is not known, though, so that the elliptical runs (3.2)-(3.3), unlike their spherical counterparts, cannot be computed from the data. Sections 3.1 and 3.2 describe two ways of obtaining sample elliptical runs and introduce the resulting tests.

3.1 Shape-based runs tests.

The most natural way to define sample elliptical runs is to replace $\mathbf{V}$ with an adequate shape estimator $\hat{\mathbf{V}}$, which leads to the *shape runs*

$$\sum_{t=2}^{n} \mathbf{U}'_{t,\boldsymbol{\theta}}(\hat{\mathbf{V}})\mathbf{U}_{t-1,\boldsymbol{\theta}}(\hat{\mathbf{V}}) \quad \text{and} \quad \sum_{t=2}^{n} \mathbf{U}_{t,\boldsymbol{\theta}}(\hat{\mathbf{V}})\mathbf{U}'_{t-1,\boldsymbol{\theta}}(\hat{\mathbf{V}}). \tag{3.4}$$

These runs are based on the so-called *standardized spatial signs* $\mathbf{U}_{t,\boldsymbol{\theta}}(\hat{\mathbf{V}})$, $t = 1, \ldots, n$, which have been used extensively in various multiple-output nonparametric problems; we refer, e.g., to Randles (2000), Taskinen et al. (2003, 2005), Larocque et al. (2007), or Hallin and Paindaveine (2008). Parallel to Marden's spherical runs in (3.1), the elliptical runs in (3.4) will clearly be sensitive to lag one serial dependence only. To improve on this, we introduce, in the same spirit as in Dufour et al. (1998), the generalized elliptical runs

$$(n-h)^{1/2}\, \hat{r}^{(n)\mathrm{M}}_{h,\boldsymbol{\theta}} := \sum_{t=h+1}^{n} \mathbf{U}'_{t,\boldsymbol{\theta}}(\hat{\mathbf{V}})\mathbf{U}_{t-h,\boldsymbol{\theta}}(\hat{\mathbf{V}}), \quad h = 1, 2, \ldots, n-1, \tag{3.5}$$

and

$$(n-h)^{1/2}\, \hat{\mathbf{r}}^{(n)\mathrm{f}}_{h,\boldsymbol{\theta}} := \sum_{t=h+1}^{n} \mathbf{U}_{t,\boldsymbol{\theta}}(\hat{\mathbf{V}})\mathbf{U}'_{t-h,\boldsymbol{\theta}}(\hat{\mathbf{V}}), \quad h = 1, 2, \ldots, n-1. \tag{3.6}$$

As in the univariate case, the corresponding runs tests reject the null of randomness when runs are "too large". For full-rank runs, which are matrix-valued statistics, sizes will be measured through the so-called Frobenius norm $\|\hat{\mathbf{r}}^{(n)\mathrm{f}}_{h,\boldsymbol{\theta}}\|^2_{\mathrm{Fr}} := \mathrm{tr}[\hat{\mathbf{r}}^{(n)\mathrm{f}}_{h,\boldsymbol{\theta}}(\hat{\mathbf{r}}^{(n)\mathrm{f}}_{h,\boldsymbol{\theta}})']$. As we will see in Section 5.1, the resulting full-rank runs test $\hat{\phi}^{(n)\mathrm{f}}_{H,\boldsymbol{\theta}}$ rejects the null of multivariate randomness $\mathcal{H}^{(n)}_{\boldsymbol{\theta}}$ at asymptotic level α as soon as

$$\begin{aligned} \hat{Q}^{(n)\mathrm{f}}_{H,\boldsymbol{\theta}} &:= k^2 \sum_{h=1}^{H} \|\hat{\mathbf{r}}^{(n)\mathrm{f}}_{h,\boldsymbol{\theta}}\|^2_{\mathrm{Fr}} \qquad (3.7)\\ &= k^2 \sum_{h=1}^{H} \frac{1}{n-h} \sum_{s,t=h+1}^{n} \mathbf{U}'_{s,\boldsymbol{\theta}}(\hat{\mathbf{V}})\mathbf{U}_{t,\boldsymbol{\theta}}(\hat{\mathbf{V}})\, \mathbf{U}'_{s-h,\boldsymbol{\theta}}(\hat{\mathbf{V}})\mathbf{U}_{t-h,\boldsymbol{\theta}}(\hat{\mathbf{V}}) > \chi^2_{k^2H,1-\alpha}, \end{aligned}$$

whereas their Marden-type counterparts $\hat{\phi}^{(n)\mathrm{M}}_{H,\boldsymbol{\theta}}$ reject the same hypothesis (still at asymptotic level α) whenever

$$\begin{aligned}\hat{Q}^{(n)\mathrm{M}}_{H,\boldsymbol{\theta}} &:= k\sum_{h=1}^{H}(\hat{r}^{(n)\mathrm{M}}_{h,\boldsymbol{\theta}})^2 \\ &= k\sum_{h=1}^{H}\frac{1}{n-h}\sum_{s,t=h+1}^{n}\mathbf{U}'_{s,\boldsymbol{\theta}}(\hat{\mathbf{V}})\mathbf{U}_{s-h,\boldsymbol{\theta}}(\hat{\mathbf{V}})\mathbf{U}'_{t,\boldsymbol{\theta}}(\hat{\mathbf{V}})\mathbf{U}_{t-h,\boldsymbol{\theta}}(\hat{\mathbf{V}}) > \chi^2_{H,1-\alpha}.\end{aligned} \tag{3.8}$$

In order to derive the asymptotic properties of these shape-based runs tests, the estimator $\hat{\mathbf{V}}$ should satisfy some assumptions including in particular root-n consistency under a broad class of (null) distributions; see Section 5.1 for details. At this point, we rather stress that the choice of $\hat{\mathbf{V}}$ is an important issue, since the multivariate runs tests above will inherit the good (or poor) robustness and equivariance/invariance properties of the shape estimator adopted. For instance, it is easy to show that, if $\hat{\mathbf{V}}$ is $\boldsymbol{\theta}$-*affine-equivariant* in the sense that

$$\hat{\mathbf{V}}(\boldsymbol{\theta}+\mathbf{A}(\mathbf{X}_1-\boldsymbol{\theta}),\ldots,\boldsymbol{\theta}+\mathbf{A}(\mathbf{X}_n-\boldsymbol{\theta})) = \frac{k\,\mathbf{A}\hat{\mathbf{V}}\mathbf{A}'}{\mathrm{tr}[\mathbf{A}\hat{\mathbf{V}}\mathbf{A}']},$$

for any $\mathbf{A}$ in the collecton GL_k of invertible $k\times k$ real matrices, then the resulting shape-based runs tests are $\boldsymbol{\theta}$-affine-invariant: any collection of samples $\{(\boldsymbol{\theta}+\mathbf{A}(\mathbf{X}_i-\boldsymbol{\theta}),\ i=1,\ldots,n) : \mathbf{A}\in GL_k\}$ leads to the same decision to reject or not the null hypothesis $\mathcal{H}^{(n)}_{\boldsymbol{\theta}}$. As explained above, achieving such invariance is important in the present context, and we therefore restrict to $\boldsymbol{\theta}$-affine-equivariant estimators of shape in the sequel.

We now list some possible choices for $\hat{\mathbf{V}}$. In the (null) model where the observations are mutually independent with a common multinormal distribution having (known) mean $\boldsymbol{\theta}$, the maximum likelihood estimator for $\mathbf{V}$ is given by $\hat{\mathbf{V}}_{\mathcal{N}} := \frac{k\,\mathbf{S}_{\boldsymbol{\theta}}}{\mathrm{tr}[\mathbf{S}_{\boldsymbol{\theta}}]}$, where $\mathbf{S}_{\boldsymbol{\theta}} := \frac{1}{n-1}\sum_{t=1}^{n}(\mathbf{X}_t-\boldsymbol{\theta})(\mathbf{X}_t-\boldsymbol{\theta})'$. This estimator, however, is root-n consistent only if the observations are i.i.d. with a common elliptical distribution admitting finite fourth-order moments, and also inherits the poor robustness properties of $\mathbf{S}_{\boldsymbol{\theta}}$. To improve on this, high breakdown point estimators of shape can be considered, such as, e.g, $\hat{\mathbf{V}}_{\mathrm{MCD}} := k\mathbf{S}_{\boldsymbol{\theta};\mathrm{MCD}}/\mathrm{tr}[\mathbf{S}_{\boldsymbol{\theta};\mathrm{MCD}}]$, where $\mathbf{S}_{\boldsymbol{\theta};\mathrm{MCD}}$ stands for the scatter matrix with smallest determinant among the $\binom{n}{\lfloor (n+1)/2 \rfloor}$ matrices $\mathbf{S}_{\boldsymbol{\theta}}$ evaluated on subseries of size $\lfloor (n+1)/2 \rfloor$. Another possible choice is the Tyler (1987) shape matrix estimator, which is defined as

$$\hat{\mathbf{V}}_{\mathrm{Tyl}} := \frac{k(\mathbf{W}'_{\boldsymbol{\theta}}\mathbf{W}_{\boldsymbol{\theta}})^{-1}}{\mathrm{tr}[(\mathbf{W}'_{\boldsymbol{\theta}}\mathbf{W}_{\boldsymbol{\theta}})^{-1}]}, \tag{3.9}$$

where $\mathbf{W}_{\boldsymbol{\theta}}$ is the (unique for $n > k(k-1)$) upper triangular $k \times k$ matrix with positive diagonal elements and a "1" in the upper left corner that satisfies

$$\frac{1}{n}\sum_{t=1}^{n}\left(\frac{\mathbf{W}_{\boldsymbol{\theta}}(\mathbf{X}_t - \boldsymbol{\theta})}{\|\mathbf{W}_{\boldsymbol{\theta}}(\mathbf{X}_t - \boldsymbol{\theta})\|}\right)\left(\frac{\mathbf{W}_{\boldsymbol{\theta}}(\mathbf{X}_t - \boldsymbol{\theta})}{\|\mathbf{W}_{\boldsymbol{\theta}}(\mathbf{X}_t - \boldsymbol{\theta})\|}\right)' = \frac{1}{k}\,\mathbf{I}_k.$$

Although Dümbgen and Tyler (2005) showed that, for "smooth" distributions, the contamination breakdown point of $\hat{\mathbf{V}}_{\mathrm{Tyl}}$ is only $1/k$, we actually favor $\hat{\mathbf{V}}_{\mathrm{Tyl}}$ over high breakdown point estimators in the sequel, mainly because it can be computed easily in any dimension k and satisfies the technical assumptions of Section 5.1 under possibly heteroskedastic observations with elliptical directions. Moreover, unlike most other shape runs, the resulting "Tyler" runs are invariant under arbitrary transformations of each $\mathbf{X}_t$ along the halfline $\boldsymbol{\theta} + \lambda(\mathbf{X}_t - \boldsymbol{\theta})$, $\lambda > 0$, hence can be regarded as sign (or directional) quantities, which is well in line with the sign nature of univariate runs.

3.2 Hyperplane-based runs tests.

The second type of sample elliptical runs we propose is based on *interdirections*, a multivariate extension of signs that was introduced by Randles (1989) in a multivariate one-sample location context. Interdirections, which have proved useful in other setups as well (see, e.g., Gieser and Randles 1997, Randles and Um 1998, Hallin and Paindaveine 2002a, Taskinen et al. 2005), are defined as follows: the interdirection $c^{(n)\mathbf{Z}}_{s,t} := c^{(n)}_{s,t}(\mathbf{Z}_1, \ldots, \mathbf{Z}_n)$ associated with $\mathbf{Z}_s$ and $\mathbf{Z}_t$ in the sequence $\mathbf{Z}_1, \ldots, \mathbf{Z}_n$ is the number of hyperplanes in $\mathbb{R}^k$ passing through the origin and $(k-1)$ out of the $(n-2)$ points $\mathbf{Z}_1, \ldots, \mathbf{Z}_{s-1}, \mathbf{Z}_{s+1}, \ldots, \mathbf{Z}_{t-1}, \mathbf{Z}_{t+1}, \ldots, \mathbf{Z}_n$, that are *separating* $\mathbf{Z}_s$ and $\mathbf{Z}_t$. The following result explains why interdirections are relevant when defining sample elliptical runs.

Lemma 3.1 *Let* $p^{(n)}_{s,t,\boldsymbol{\theta}} := c^{(n)}_{s,t,\boldsymbol{\theta}} / \binom{n-2}{k-1}$, *where* $c^{(n)}_{s,t,\boldsymbol{\theta}}$ *denotes the interdirection associated with* $\mathbf{X}_s - \boldsymbol{\theta}$ *and* $\mathbf{X}_t - \boldsymbol{\theta}$ *in the* k*-variate series* $\mathbf{X}_1 - \boldsymbol{\theta}, \ldots, \mathbf{X}_n - \boldsymbol{\theta}$. *Then, under* $\mathcal{H}^{(n)}_{\boldsymbol{\theta},\mathbf{V}}$, $\cos(\pi p^{(n)}_{s,t,\boldsymbol{\theta}}) - \mathbf{U}'_{s,\boldsymbol{\theta}}(\mathbf{V})\mathbf{U}_{t,\boldsymbol{\theta}}(\mathbf{V}) \to 0$ *in quadratic mean as* $n \to \infty$.

This result suggests that we regard

$$(n-h)^{1/2}\, \tilde{r}^{(n)\mathrm{M}}_{h,\boldsymbol{\theta}} := \sum_{t=h+1}^{n} \cos(\pi p^{(n)}_{t,t-h,\boldsymbol{\theta}}) \tag{3.10}$$

as a hyperplane-based version of the sample runs in (3.5). Although there is no hyperplane-based version of the full-rank runs in (3.6), it is easy to define a hyperplane-based version of the statistic $\hat{Q}^{(n)\mathrm{f}}_{H,\boldsymbol{\theta}}$ since the latter involves the observations through inner products of standardized spatial signs only. More precisely, the hyperplane-based version $\tilde{\phi}^{(n)\mathrm{f}}_{H,\boldsymbol{\theta}}$ of the test $\hat{\phi}^{(n)\mathrm{f}}_{H,\boldsymbol{\theta}}$ rejects the null hypothesis $\mathcal{H}^{(n)}_{\boldsymbol{\theta}}$ at asymptotic level α whenever

$$\tilde{Q}^{(n)\mathrm{f}}_{H,\boldsymbol{\theta}} := k^2 \sum_{h=1}^{H} \frac{1}{n-h} \sum_{s,t=h+1}^{n} \cos(\pi p^{(n)}_{s,t,\boldsymbol{\theta}})\, \cos(\pi p^{(n)}_{s-h,t-h,\boldsymbol{\theta}}) > \chi^2_{k^2 H, 1-\alpha}$$

(hence actually coincides with the sign test proposed in Hallin and Paindaveine (2002b)), whereas, of course, the hyperplane-based Marden runs test $\tilde{\phi}^{(n)\mathrm{M}}_{H,\boldsymbol{\theta}}$ rejects the same hypothesis (still at asymptotic level α) as soon as

$$\tilde{Q}^{(n)\mathrm{M}}_{H,\boldsymbol{\theta}} := k \sum_{h=1}^{H} (\tilde{r}^{(n)\mathrm{M}}_{h,\boldsymbol{\theta}})^2 = k \sum_{h=1}^{H} \frac{1}{n-h} \sum_{s,t=h+1}^{n} \cos(\pi p^{(n)}_{s,s-h,\boldsymbol{\theta}}) \cos(\pi p^{(n)}_{t,t-h,\boldsymbol{\theta}}) > \chi^2_{H,1-\alpha}.$$

As shown in Randles (1989), interdirections are affine-invariant in the sense that $c^{(n)}_{s,t}(\mathbf{A}\mathbf{Z}_1, \ldots, \mathbf{A}\mathbf{Z}_n) = c^{(n)}_{s,t}(\mathbf{Z}_1, \ldots, \mathbf{Z}_n)$ for any $\mathbf{A} \in GL_k$; $\boldsymbol{\theta}$-affine-invariance of the hyperplane-based runs in (3.10), and of the hyperplane-based runs tests $\tilde{\phi}^{(n)\mathrm{f}}_{H,\boldsymbol{\theta}}$ and $\tilde{\phi}^{(n)\mathrm{M}}_{H,\boldsymbol{\theta}}$

above, directly follows. Note that these hyperplane-based statistics may also be regarded as *sign* statistics, since the $p^{(n)}_{s,t,\boldsymbol{\theta}}$'s are invariant under arbitrary *radial* transformations of each observation $\mathbf{X}_t$ along the halfline $\boldsymbol{\theta} + \lambda(\mathbf{X}_t - \boldsymbol{\theta})$, $\lambda > 0$.

Since $\hat{\phi}^{(n)\mathrm{f}}_{H,\boldsymbol{\theta}}$, $\tilde{\phi}^{(n)\mathrm{f}}_{H,\boldsymbol{\theta}}$, $\hat{\phi}^{(n)\mathrm{M}}_{H,\boldsymbol{\theta}}$, and $\tilde{\phi}^{(n)\mathrm{M}}_{H,\boldsymbol{\theta}}$ reduce to the classical runs test for $k = 1$, they may all be regarded as multivariate runs tests. Note that, as a corollary of the radial- and affine-invariance, $\tilde{\phi}^{(n)\mathrm{f}}_{H,\boldsymbol{\theta}}/\tilde{\phi}^{(n)\mathrm{M}}_{H,\boldsymbol{\theta}}$—and $\hat{\phi}^{(n)\mathrm{f}}_{H,\boldsymbol{\theta}}/\hat{\phi}^{(n)\mathrm{M}}_{H,\boldsymbol{\theta}}$, provided that they are based on the shape estimator $\hat{\mathbf{V}}_{\mathrm{Tyl}}$—are strictly distribution-free under $\mathcal{H}^{(n)}_{\boldsymbol{\theta}}$.

3.3 The $\boldsymbol{\theta}$-unspecified case.

Clearly, in order to turn the proposed $\boldsymbol{\theta}$-specified runs procedures into tests for the null of randomness $\mathcal{H}^{(n)} := \bigcup_{\boldsymbol{\theta}\in\mathbb{R}^k} \bigcup_{\mathbf{V}\in\mathcal{M}_k} \mathcal{H}^{(n)}_{\boldsymbol{\theta},\mathbf{V}}$ under which the center $\boldsymbol{\theta}$ remains unspecified (see Section 2), an estimate of $\boldsymbol{\theta}$—even an estimate of $(\boldsymbol{\theta}, \mathbf{V})$ if one wants to avoid hyperplane-based procedures—is needed. We propose using the Hettmansperger and Randles (2002) estimator $(\hat{\boldsymbol{\theta}}_{\mathrm{HR}}, \hat{\mathbf{V}}_{\mathrm{HR}})$, which is implicitly defined as the solution of the simultaneous M-equations (with $\hat{\mathbf{V}}_{\mathrm{HR}}$ still normalized so that $\mathrm{tr}[\hat{\mathbf{V}}_{\mathrm{HR}}] = k$)

$$\frac{1}{n}\sum_{t=1}^{n} \mathbf{U}_{t,\hat{\boldsymbol{\theta}}_{\mathrm{HR}}}(\hat{\mathbf{V}}_{\mathrm{HR}}) = \mathbf{0} \quad \text{and} \quad \frac{1}{n}\sum_{t=1}^{n} \mathbf{U}_{t,\hat{\boldsymbol{\theta}}_{\mathrm{HR}}}(\hat{\mathbf{V}}_{\mathrm{HR}})\mathbf{U}'_{t,\hat{\boldsymbol{\theta}}_{\mathrm{HR}}}(\hat{\mathbf{V}}_{\mathrm{HR}}) = \frac{1}{k}\mathbf{I}_k. \tag{3.11}$$

This estimator is root-n consistent under the whole null hypothesis $\mathcal{H}^{(n)}$, since it does not require strict ellipticity, identically distributed observations, nor any finite moment assumptions. Moreover, $(\hat{\boldsymbol{\theta}}_{\mathrm{HR}}, \hat{\mathbf{V}}_{\mathrm{HR}})$ is affine-equivariant in the sense that, for any $\mathbf{A} \in GL_k$ and any k-vector $\mathbf{b}$, $\hat{\boldsymbol{\theta}}_{\mathrm{HR};\mathbf{A},\mathbf{b}} = \mathbf{A}\hat{\boldsymbol{\theta}}_{\mathrm{HR}} + \mathbf{b}$ and $\hat{\mathbf{V}}_{\mathrm{HR};\mathbf{A},\mathbf{b}} = \frac{k\,\mathbf{A}\hat{\mathbf{V}}_{\mathrm{HR}}\mathbf{A}'}{\mathrm{tr}[\mathbf{A}\hat{\mathbf{V}}_{\mathrm{HR}}\mathbf{A}']}$, where $\hat{\boldsymbol{\theta}}_{\mathrm{HR};\mathbf{A},\mathbf{b}}$ (resp., $\hat{\mathbf{V}}_{\mathrm{HR};\mathbf{A},\mathbf{b}}$) stands for $\hat{\boldsymbol{\theta}}_{\mathrm{HR}}$ (resp., $\hat{\mathbf{V}}_{\mathrm{HR}}$) evaluated at $\mathbf{A}\mathbf{X}_1 + \mathbf{b}, \ldots, \mathbf{A}\mathbf{X}_n + \mathbf{b}$. Finally, $\hat{\boldsymbol{\theta}}_{\mathrm{HR}}$ is an extension of the sample median to multivariate data, hence is well in line with the sign nature of univariate runs.

The resulting shape-based Marden runs tests $\hat{\phi}^{(n)\mathrm{M}}_{H}$ (resp., hyperplane-based Marden runs tests $\tilde{\phi}^{(n)\mathrm{M}}_{H}$) reject $\mathcal{H}^{(n)}$ at asymptotic level α whenever $\hat{Q}^{(n)\mathrm{M}}_{H} := \hat{Q}^{(n)\mathrm{M}}_{H,\hat{\boldsymbol{\theta}}_{\mathrm{HR}}}$ (resp., $\tilde{Q}^{(n)\mathrm{M}}_{H} := \tilde{Q}^{(n)\mathrm{M}}_{H,\hat{\boldsymbol{\theta}}_{\mathrm{HR}}}$), where the shape estimator $\hat{\mathbf{V}}_{\mathrm{HR}}$ is used, exceeds $\chi^2_{H,1-\alpha}$.

Similarly, full-rank runs tests for $\mathcal{H}^{(n)}$, in their shape-based version $\hat{\phi}_H^{(n)\text{f}}$ (resp., hyperplane-based version $\tilde{\phi}_H^{(n)\text{f}}$) reject the null at asymptotic level α whenever $\hat{Q}_H^{(n)\text{f}} := \hat{Q}_{H,\hat{\boldsymbol{\theta}}_{\text{HR}}}^{(n)\text{f}}$ (resp., $\tilde{Q}_H^{(n)\text{f}} := \tilde{Q}_{H,\hat{\boldsymbol{\theta}}_{\text{HR}}}^{(n)\text{f}}$) exceeds $\chi^2_{k^2H,1-\alpha}$. Note that $\tilde{\phi}_H^{(n)\text{M}}/\tilde{\phi}_H^{(n)\text{f}}$ do not require any shape estimate. Finally, the affine-equivariance of $(\hat{\boldsymbol{\theta}}_{\text{HR}}, \hat{\mathbf{V}}_{\text{HR}})$ entails that these four runs test statistics are affine-invariant, in the sense that their values remain unchanged if a common arbitrary invertible affine transformation is applied to the n observations.

4 A REAL DATA EXAMPLE.

In this section, we consider a series of daily closing prices for $k = 4$ major European stock indices: Germany DAX, Switzerland SMI, France CAC, and UK FTSE. The data, available under the name *EuStockMarkets* in current distributions of the R software, are sampled in business time (i.e., weekends and holidays are omitted), yielding $1,860$ closing prices for each index between 1991 and 1998. We study the series of the $n = 1,859$ four-dimensional resulting log returns; see Figure 1. The scatter plot of SIM vs DAX log returns displayed in the same figure is not of an elliptical nature since negative returns are more extreme than positive ones; this, however, is fully compatible with the assumption that the joint distribution has elliptical directions. This skewness and the obvious fat tails suggest that our runs tests are well adapted to this data set.

Table 1 reports the p-values of the $\boldsymbol{\theta}$-unspecified shape-based full-rank runs tests $\hat{\phi}_H^{(n)\text{f}}$ and of their Marden analogs $\hat{\phi}_H^{(n)\text{M}}$, $H = 1, 2, 3$, for (i) the series of log returns and (ii) the series of squared log returns. For log returns, it also makes sense to consider the corresponding $\boldsymbol{\theta} = \mathbf{0}$-specified tests $\hat{\phi}_{H,\boldsymbol{\theta}}^{(n)\text{M}}$ and $\hat{\phi}_{H,\boldsymbol{\theta}}^{(n)\text{f}}$. Of course, hyperplane-based runs tests cannot be implemented for such a large sample size (still, they are of interest for small dimensions k and small-to-moderate sample sizes, as is illustrated in the

Monte-Carlo study of Section 6). For the sake of comparison, we also include p-values of Gaussian Portmanteau tests based on classical autocorrelation matrices; we denote their $\boldsymbol{\theta}$-specified and $\boldsymbol{\theta}$-unspecified versions by $\phi_{H,\boldsymbol{\theta}}^{(n)\mathcal{N}}$ and $\phi_{H}^{(n)\mathcal{N}}$, respectively. These highly non-robust procedures, which are described in much detail in Section 5.3, are also used there as benchmarks for efficiency comparisons.

Inspection of the p-values in Table 1 shows that, as expected, all tests lead to rejection of the null of randomness at any standard level, with the only exception, for squared log returns, of $\hat{\phi}_1^{(n)\mathrm{M}}$. Quite interestingly, the fact that $\hat{\phi}_1^{(n)\mathrm{M}}$ does not reject the null allows to get some insight about why rejection occurs for $\hat{\phi}_1^{(n)\mathrm{f}}$. In the present example, this actually reveals, as we will see in Section 5.2, that no index strongly determine their own future volatility while there are some indices whose volatility is mainly driven by other indices (we delay details on this to Section 5.2 since it requires an investigation of the asymptotic nonnull behaviors of these runs tests).

To assess the robustness of the results above and to illustrate the use of hyperplane-based tests on this real data set, we recorded, for each of the 80 subseries of length 100 associated with time indices $(s, s+1, \ldots, s+99)$, $s = 1, \ldots, 80$, the p-values of the same tests as in Table 1, along with those of the corresponding hyperplane-based runs tests; see Figure 2. It is readily seen that Gaussian Portmanteau tests are extremely sensitive to large returns, particularly so for squared log returns where their p-value jumps from .0001 to .9956 at $s = 36$, that is, exactly when the very extreme negative log return at $t = 35$ (which is visible in the lower left corner of the scatter plot in Figure 1) leaves the subseries of interest. Comparatively, our multivariate runs tests exhibit a much more stable behavior. Eventually, note that, for the present data, hyperplane-based runs tests behave essentially as their shape-based counterparts.

5 ASYMPTOTIC BEHAVIOR.

5.1 Asymptotic null behavior.

The following assumptions on $\hat{\mathbf{V}}$ are needed to derive the asymptotic properties of the $\boldsymbol{\theta}$-specified shape-based runs tests (for the sake of simplicity, we will only consider the versions of the $\boldsymbol{\theta}$-unspecified tests based on the estimators $(\hat{\boldsymbol{\theta}}_{\mathrm{HR}}, \hat{\mathbf{V}}_{\mathrm{HR}})$ of Section 3.3).

Assumption $(\mathrm{A}_{\boldsymbol{\theta}})$. *The sequence of estimators* $\hat{\mathbf{V}} = \hat{\mathbf{V}}(\mathbf{X}_1, \ldots, \mathbf{X}_n)$, $n \in \mathbb{N}_0$, *is (i) root-n consistent: for any* $\mathbf{V} \in \mathcal{M}_k$, $n^{1/2}(\hat{\mathbf{V}} - \mathbf{V}) = O_{\mathrm{P}}(1)$ *as* $n \to \infty$, *under* $\mathcal{H}^{(n)}_{\boldsymbol{\theta},\mathbf{V}}$; *(ii) invariant under reflections about* $\boldsymbol{\theta}$: $\hat{\mathbf{V}}(\boldsymbol{\theta} + s_1(\mathbf{X}_1 - \boldsymbol{\theta}), \ldots, \boldsymbol{\theta} + s_n(\mathbf{X}_n - \boldsymbol{\theta})) = \hat{\mathbf{V}}(\mathbf{X}_1, \ldots, \mathbf{X}_n)$ *for any* $s_1, \ldots, s_n \in \{-1, 1\}$; *(iii) invariant under permutations:* $\hat{\mathbf{V}}(\mathbf{X}_{\xi(1)}, \ldots, \mathbf{X}_{\xi(n)}) = \hat{\mathbf{V}}(\mathbf{X}_1, \ldots, \mathbf{X}_n)$ *for any permutation* ξ *of* $\{1, \ldots, n\}$.

We first study the asymptotic null behavior of $\hat{\mathbf{R}}^{(n)\mathrm{M}}_{H,\boldsymbol{\theta}} := (\hat{r}^{(n)\mathrm{M}}_{1,\boldsymbol{\theta}}, \ldots, \hat{r}^{(n)\mathrm{M}}_{H,\boldsymbol{\theta}})'$, $\tilde{\mathbf{R}}^{(n)\mathrm{M}}_{H,\boldsymbol{\theta}} := (\tilde{r}^{(n)\mathrm{M}}_{1,\boldsymbol{\theta}}, \ldots, \tilde{r}^{(n)\mathrm{M}}_{H,\boldsymbol{\theta}})'$, and $\hat{\mathbf{R}}^{(n)\mathrm{f}}_{H,\boldsymbol{\theta}} := ((\mathrm{vec}\, \hat{\mathbf{r}}^{(n)\mathrm{f}}_{1,\boldsymbol{\theta}})', \ldots, (\mathrm{vec}\, \hat{\mathbf{r}}^{(n)\mathrm{f}}_{H,\boldsymbol{\theta}})')'$, where $\mathrm{vec}\, \mathbf{C}$ stacks the columns of the array $\mathbf{C}$ on top of each other.

Lemma 5.1 *Fix* $\boldsymbol{\theta} \in \mathbb{R}^k$, $\mathbf{V} \in \mathcal{M}_k$, *and* $H \in \mathbb{N}_0$, *and let Assumption* $(A_{\boldsymbol{\theta}})$ *hold. Define* $\mathbf{R}^{(n)\mathrm{M}}_{H,\boldsymbol{\theta};\mathbf{V}} := (r^{(n)\mathrm{M}}_{1,\boldsymbol{\theta};\mathbf{V}}, \ldots, r^{(n)\mathrm{M}}_{H,\boldsymbol{\theta};\mathbf{V}})'$ *and* $\mathbf{R}^{(n)\mathrm{f}}_{H,\boldsymbol{\theta};\mathbf{V}} := ((\mathrm{vec}\, \mathbf{r}^{(n)\mathrm{f}}_{1,\boldsymbol{\theta};\mathbf{V}})', \ldots, (\mathrm{vec}\, \mathbf{r}^{(n)\mathrm{f}}_{H,\boldsymbol{\theta};\mathbf{V}})')'$, *where we let* $r^{(n)\mathrm{M}}_{h,\boldsymbol{\theta};\mathbf{V}} := (n-h)^{-1/2} \sum_{t=h+1}^{n} \mathbf{U}'_{t,\boldsymbol{\theta}}(\mathbf{V}) \mathbf{U}_{t-h,\boldsymbol{\theta}}(\mathbf{V})$ *and* $(n-h)^{1/2}\, \mathbf{r}^{(n)\mathrm{f}}_{h,\boldsymbol{\theta};\mathbf{V}} := \sum_{t=h+1}^{n} \mathbf{U}_{t,\boldsymbol{\theta}}(\mathbf{V}) \mathbf{U}'_{t-h,\boldsymbol{\theta}}(\mathbf{V})$. *Then, (i) under* $\mathcal{H}^{(n)}_{\boldsymbol{\theta},\mathbf{V}}$,

$$\mathrm{E}[\|\hat{\mathbf{R}}^{(n)\mathrm{M}}_{H,\boldsymbol{\theta}} - \mathbf{R}^{(n)\mathrm{M}}_{H,\boldsymbol{\theta};\mathbf{V}}\|^2], \quad \mathrm{E}[\|\tilde{\mathbf{R}}^{(n)\mathrm{M}}_{H,\boldsymbol{\theta}} - \mathbf{R}^{(n)\mathrm{M}}_{H,\boldsymbol{\theta};\mathbf{V}}\|^2], \quad \textit{and} \quad \mathrm{E}[\|\hat{\mathbf{R}}^{(n)\mathrm{f}}_{H,\boldsymbol{\theta}} - \mathbf{R}^{(n)\mathrm{f}}_{H,\boldsymbol{\theta};\mathbf{V}}\|^2]$$

are $o(1)$ *as* $n \to \infty$. *(ii) under* $\mathcal{H}^{(n)}_{\boldsymbol{\theta}}$, *both* $\hat{\mathbf{R}}^{(n)\mathrm{M}}_{H,\boldsymbol{\theta}}$ *and* $\tilde{\mathbf{R}}^{(n)\mathrm{M}}_{H,\boldsymbol{\theta}}$ *are asymptotically normal with mean zero and covariance matrix* $\frac{1}{k}\mathbf{I}_H$, *whereas* $\hat{\mathbf{R}}^{(n)\mathrm{f}}_{H,\boldsymbol{\theta}}$ *is asymptotically normal with mean zero and covariance matrix* $\frac{1}{k^2}\mathbf{I}_{k^2 H}$.

Note that Lemma 5.1(i) entails that, under $\mathcal{H}^{(n)}_{\boldsymbol{\theta},\mathbf{V}}$, $\hat{Q}^{(n)\mathrm{M}}_{H,\boldsymbol{\theta}} = k\|\hat{\mathbf{R}}^{(n)\mathrm{M}}_{H,\boldsymbol{\theta}}\|^2 = k\|\mathbf{R}^{(n)\mathrm{M}}_{H,\boldsymbol{\theta};\mathbf{V}}\|^2 + o_{\mathrm{P}}(1)$ and $\tilde{Q}^{(n)\mathrm{M}}_{H,\boldsymbol{\theta}} = k\|\tilde{\mathbf{R}}^{(n)\mathrm{M}}_{H,\boldsymbol{\theta}}\|^2 = k\|\mathbf{R}^{(n)\mathrm{M}}_{H,\boldsymbol{\theta};\mathbf{V}}\|^2 + o_{\mathrm{P}}(1)$ as $n \to \infty$—which implies that $\hat{Q}^{(n)\mathrm{M}}_{H,\boldsymbol{\theta}}$ and $\tilde{Q}^{(n)\mathrm{M}}_{H,\boldsymbol{\theta}}$ are asymptotically equivalent under the null (hence also, from contiguity, under the sequences of local alternatives we introduce in Section 5.2). Lemma 5.1(ii) then yields that both $\hat{Q}^{(n)\mathrm{M}}_{H,\boldsymbol{\theta}}$ and $\tilde{Q}^{(n)\mathrm{M}}_{H,\boldsymbol{\theta}}$ are asymptotically χ^2_H under $\mathcal{H}^{(n)}_{\boldsymbol{\theta}}$, which implies that the multivariate runs tests $\hat{\phi}^{(n)\mathrm{M}}_{H,\boldsymbol{\theta}}$ and $\tilde{\phi}^{(n)\mathrm{M}}_{H,\boldsymbol{\theta}}$ from Section 3 indeed have asymptotic level α. Similarly, Lemma 5.1 entails that the full-rank runs test $\hat{\phi}^{(n)\mathrm{f}}_{H,\boldsymbol{\theta}}$ has asymptotic level α since it shows that $\hat{Q}^{(n)\mathrm{f}}_{H,\boldsymbol{\theta}} = k^2\, \|\hat{\mathbf{R}}^{(n)\mathrm{f}}_{H,\boldsymbol{\theta}}\|^2$ is asymptotically $\chi^2_{k^2H}$ under the null. As for $\hat{\phi}^{(n)\mathrm{f}}_{H,\boldsymbol{\theta}}$, the result follows from the fact that it was shown in Hallin and Paindaveine (2002b) that $\tilde{Q}^{(n)\mathrm{f}}_{H,\boldsymbol{\theta}} = k^2\, \|\mathbf{R}^{(n)\mathrm{f}}_{H,\boldsymbol{\theta};\mathbf{V}}\|^2 + o_{\mathrm{P}}(1)$ as $n \to \infty$ under $\mathcal{H}^{(n)}_{\boldsymbol{\theta},\mathbf{V}}$ (their proof, which was derived under strictly elliptical assumptions, trivially holds in the present broader setup).

Turning then to the $\boldsymbol{\theta}$-unspecified runs tests, we define the multivariate shape-based runs $\hat{\mathbf{R}}^{(n)\mathrm{M}}_{H}$ and $\hat{\mathbf{R}}^{(n)\mathrm{f}}_{H}$ by replacing $(\boldsymbol{\theta}, \mathbf{V})$ with $(\hat{\boldsymbol{\theta}}_{\mathrm{HR}}, \hat{\mathbf{V}}_{\mathrm{HR}})$ (see Section 3.3) in $\mathbf{R}^{(n)\mathrm{M}}_{H,\boldsymbol{\theta};\mathbf{V}}$ and $\mathbf{R}^{(n)\mathrm{f}}_{H,\boldsymbol{\theta};\mathbf{V}}$, respectively. Similarly, multivariate hyperplane-based Marden runs, which do not require any estimate for $\mathbf{V}$, are simply defined as $\tilde{\mathbf{R}}^{(n)\mathrm{M}}_{H} := \tilde{\mathbf{R}}^{(n)\mathrm{M}}_{H,\hat{\boldsymbol{\theta}}_{\mathrm{HR}}}$. We then have the following result.

Proposition 5.1 *Fix $\boldsymbol{\theta} \in \mathbb{R}^k$, $\mathbf{V} \in \mathcal{M}_k$, and $H \in \mathbb{N}_0$. Then, (i) under $\mathcal{H}^{(n)}_{\boldsymbol{\theta},\mathbf{V}}$, $\hat{\mathbf{R}}^{(n)\mathrm{M}}_{H} - \mathbf{R}^{(n)\mathrm{M}}_{H,\boldsymbol{\theta};\mathbf{V}}$, $\tilde{\mathbf{R}}^{(n)\mathrm{M}}_{H} - \mathbf{R}^{(n)\mathrm{M}}_{H,\boldsymbol{\theta};\mathbf{V}}$, and $\hat{\mathbf{R}}^{(n)\mathrm{f}}_{H} - \mathbf{R}^{(n)\mathrm{f}}_{H,\boldsymbol{\theta};\mathbf{V}}$ are $o_{\mathrm{P}}(1)$ as $n \to \infty$; (ii) under $\mathcal{H}^{(n)} = \bigcup_{\boldsymbol{\theta}\in\mathbb{R}^k} \bigcup_{\mathbf{V}\in\mathcal{M}_k} \mathcal{H}^{(n)}_{\boldsymbol{\theta},\mathbf{V}}$, $\hat{Q}^{(n)\mathrm{M}}_{H}$ and $\tilde{Q}^{(n)\mathrm{M}}_{H}$ (resp., $\hat{Q}^{(n)\mathrm{f}}_{H}$ and $\tilde{Q}^{(n)\mathrm{f}}_{H}$) are asymptotically chi-square with H (resp., k^2H) degrees of freedom.*

The $\boldsymbol{\theta}$-unspecified runs tests introduced in Section 3.3 therefore all have asymptotic level α under the null of randomness $\mathcal{H}^{(n)}$.

5.2 ULAN and local asymptotic powers.

In this section, we study the asymptotic powers of the proposed runs tests against the following sequences of local vector autoregressive moving average (VARMA) alternatives. Write $\mathrm{P}^{(n)}_{\boldsymbol{\vartheta},\mathbf{V},f}$ for the hypothesis under which the series $\mathbf{X}_1, \ldots, \mathbf{X}_n$ is a finite realization of the elliptical VARMA(p, q) process

$$(\mathbf{X}_t - \boldsymbol{\theta}) - \sum_{r=1}^{p} \mathbf{A}_r (\mathbf{X}_{t-r} - \boldsymbol{\theta}) = \boldsymbol{\varepsilon}_t - \sum_{s=1}^{q} \mathbf{B}_s \boldsymbol{\varepsilon}_{t-s}, \quad t \in \mathbb{Z},$$

where $\boldsymbol{\vartheta} := (\boldsymbol{\theta}', (\mathrm{vec}\,\mathbf{A}_1)', \ldots, (\mathrm{vec}\,\mathbf{A}_p)', (\mathrm{vec}\,\mathbf{B}_1)', \ldots, (\mathrm{vec}\,\mathbf{B}_q)')' \in \mathbb{R}^L := \mathbb{R}^{k+k^2(p+q)}$ collects the location center and the vectorized forms of the $k \times k$ matrices $\mathbf{A}_1, \ldots, \mathbf{A}_p$, $\mathbf{B}_1, \ldots, \mathbf{B}_q$, and where the (absolutely continuous) random k-vectors $\boldsymbol{\varepsilon}_t$, $t \in \mathbb{Z}$, are i.i.d. with the elliptically symmetric pdf in (2.2). With this notation, the local alternatives we consider are of the form $\mathrm{P}^{(n)}_{(\boldsymbol{\theta}', \mathbf{0}')' + n^{-1/2}\boldsymbol{\tau}^{(n)}, \mathbf{V}, f}$, where $\boldsymbol{\tau}^{(n)} := (\mathbf{t}^{(n)\prime}, (\mathrm{vec}\,\mathbf{A}_1^{(n)})'$, $\ldots, (\mathrm{vec}\,\mathbf{A}_p^{(n)})', (\mathrm{vec}\,\mathbf{B}_1^{(n)})', \ldots, (\mathrm{vec}\,\mathbf{B}_q^{(n)})')'$ is a bounded sequence, and where the radial density f satisfies some mild regularity conditions ensuring that the family of probability distributions $\{\mathrm{P}^{(n)}_{\boldsymbol{\vartheta}, \mathbf{V}, f} : \boldsymbol{\vartheta} \in \mathbb{R}^L\}$ is *uniformly locally and asymptotically normal* (ULAN) at any parameter value of the form $\boldsymbol{\vartheta} = (\boldsymbol{\theta}', \mathbf{0}')'$. More precisely, we assume that f belongs to the collection $\mathcal{F}_{\mathrm{ULAN}}$ of absolutely continuous radial densities for which, denoting by f' the a.e.-derivative of f and letting $\varphi_f := -f'/f$, the integrals $\mu_{k+1,f} = \int_0^\infty r^{k+1} f(r)\, dr$ and $\mathcal{I}_{k,f} := (\mu_{k-1,f})^{-1} \int_0^\infty \varphi_f^2(r)\, r^{k-1} f(r)\, dr$ are finite. Letting $\pi := \max(p, q)$ and defining the $k^2\pi \times k^2(p+q)$ matrix $\mathbf{M}$ as

$$\mathbf{M} := \begin{pmatrix} \mathbf{I}_{k^2 p} & \mathbf{I}_{k^2 q} \\ \mathbf{0}_{k^2(\pi - p) \times (k^2 p)} & \mathbf{0}_{k^2(\pi - q) \times (k^2 q)} \end{pmatrix},$$

we then have the following ULAN result.

Proposition 5.2 *Fix $\mathbf{V} \in \mathcal{M}_k$ and $f \in \mathcal{F}_{\mathrm{ULAN}}$. Then, writing $\boldsymbol{\gamma}^{(n)}_{h, \boldsymbol{\theta}, \mathbf{V}, f} := \frac{1}{n-h}\, \mathbf{V}^{-1/2} \sum_{t=h+1}^{n} \varphi_f(d_{t,\boldsymbol{\theta}}(\mathbf{V}))\, d_{t-h,\boldsymbol{\theta}}(\mathbf{V})\, \mathbf{U}_{t,\boldsymbol{\theta}}(\mathbf{V}) \mathbf{U}'_{t-h,\boldsymbol{\theta}}(\mathbf{V})\, \mathbf{V}^{1/2}$, the family $\{\mathrm{P}^{(n)}_{\boldsymbol{\vartheta}, \mathbf{V}, f} : \boldsymbol{\vartheta} \in \mathbb{R}^L\}$ is ULAN at any $\boldsymbol{\vartheta} = (\boldsymbol{\theta}', \mathbf{0}')'$, with central sequence $\boldsymbol{\Delta}^{(n)}_{\boldsymbol{\vartheta}, \mathbf{V}, f} := ((\boldsymbol{\Delta}^{I(n)}_{\boldsymbol{\vartheta}, \mathbf{V}, f})', (\boldsymbol{\Delta}^{II(n)}_{\boldsymbol{\vartheta}, \mathbf{V}, f})')'$, where $\boldsymbol{\Delta}^{I(n)}_{\boldsymbol{\vartheta}, \mathbf{V}, f} := n^{-1/2}\, \mathbf{V}^{-1/2} \sum_{t=1}^{n} \varphi_f(d_{t,\boldsymbol{\theta}}(\mathbf{V}))\, \mathbf{U}_{t,\boldsymbol{\theta}}(\mathbf{V})$ and*

$$\boldsymbol{\Delta}^{II(n)}_{\boldsymbol{\vartheta}, \mathbf{V}, f} := \mathbf{M}' \Big((n-1)^{1/2}\, (\mathrm{vec}\, \boldsymbol{\gamma}^{(n)}_{1, \boldsymbol{\theta}, \mathbf{V}, f})', \ldots, (n-\pi)^{1/2}\, (\mathrm{vec}\, \boldsymbol{\gamma}^{(n)}_{\pi, \boldsymbol{\theta}, \mathbf{V}, f})' \Big)',$$

and information matrix

$$\boldsymbol{\Gamma}_{\boldsymbol{\vartheta},\mathbf{V},f} := \begin{pmatrix} \boldsymbol{\Gamma}^{I}_{\boldsymbol{\vartheta},\mathbf{V},f} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Gamma}^{II}_{\boldsymbol{\vartheta},\mathbf{V},f} \end{pmatrix} := \begin{pmatrix} \frac{\mathcal{I}_{k,f}}{k}\mathbf{V}^{-1} & \mathbf{0} \\ \mathbf{0} & \frac{\mu_{k+1,f}\,\mathcal{I}_{k,f}}{k^2\,\mu_{k-1,f}}\,\mathbf{M}'[\mathbf{I}_\pi \otimes (\mathbf{V}\otimes\mathbf{V}^{-1})]\mathbf{M} \end{pmatrix}.$$

More precisely, for any $\boldsymbol{\vartheta}^{(n)} = (\boldsymbol{\theta}^{(n)\prime}, \mathbf{0}')'$ *and* $\boldsymbol{\vartheta} = (\boldsymbol{\theta}', \mathbf{0}')'$ *with* $\boldsymbol{\theta}^{(n)} = \boldsymbol{\theta} + O(n^{-1/2})$, *we have that (i) under* $\mathrm{P}^{(n)}_{\boldsymbol{\vartheta}^{(n)},\mathbf{V},f}$,

$$\log\left(\frac{d\mathrm{P}^{(n)}_{\boldsymbol{\vartheta}^{(n)}+n^{-1/2}\boldsymbol{\tau}^{(n)},\mathbf{V},f}}{d\mathrm{P}^{(n)}_{\boldsymbol{\vartheta}^{(n)},\mathbf{V},f}}\right) = (\boldsymbol{\tau}^{(n)})'\boldsymbol{\Delta}^{(n)}_{\boldsymbol{\vartheta}^{(n)},\mathbf{V},f} - \frac{1}{2}(\boldsymbol{\tau}^{(n)})'\boldsymbol{\Gamma}_{\boldsymbol{\vartheta},\mathbf{V},f}\,\boldsymbol{\tau}^{(n)} + o_{\mathrm{P}}(1)$$

as $n\to\infty$, *and that (ii)* $\boldsymbol{\Delta}^{(n)}_{\boldsymbol{\vartheta}^{(n)},\mathbf{V},f}$, *still under* $\mathrm{P}^{(n)}_{\boldsymbol{\vartheta}^{(n)},\mathbf{V},f}$, *is asymptotically* $\mathcal{N}(\mathbf{0}, \boldsymbol{\Gamma}_{\boldsymbol{\vartheta},\mathbf{V},f})$.

The nonnull asymptotic distributions of the proposed multivariate runs test statistics, under the sequences of local alternatives described above, hence also the asymptotic local powers of the corresponding tests, quite easily follow from this ULAN property and a standard application of Le Cam's third Lemma.

Proposition 5.3 *Fix* $\boldsymbol{\vartheta} = (\boldsymbol{\theta}', \mathbf{0}')' \in \mathbb{R}^L$, $\mathbf{V} \in \mathcal{M}_k$, $f \in \mathcal{F}_{\mathrm{ULAN}}$, *and* $H \in \mathbb{N}_0$, *and define* $\zeta_{k,f} := \frac{\mu_{k,f}\xi_{k,f}}{k\,\mu_{k-1,f}}$, *where* $\xi_{k,f} := (\mu_{k-1,f})^{-1}\int_0^\infty \varphi_f(r)\,r^{k-1}f(r)\,dr$. *Then, letting Assumption* ($A_{\boldsymbol{\theta}}$) *hold for results on* $\boldsymbol{\theta}$*-specified shape-based quantities, we have that (i) under* $\{\mathrm{P}^{(n)}_{\boldsymbol{\vartheta}+n^{-1/2}\boldsymbol{\tau}^{(n)},\mathbf{V},f}\}$,

$$\hat{\mathbf{R}}^{(n)\mathrm{M}}_{H,\boldsymbol{\theta}} - \frac{\zeta_{k,f}}{k}\begin{pmatrix} \mathrm{tr}[\mathbf{A}^{(n)}_1 - \mathbf{B}^{(n)}_1] \\ \vdots \\ \mathrm{tr}[\mathbf{A}^{(n)}_H - \mathbf{B}^{(n)}_H] \end{pmatrix} \quad \textit{and} \quad \tilde{\mathbf{R}}^{(n)\mathrm{M}}_{H,\boldsymbol{\theta}} - \frac{\zeta_{k,f}}{k}\begin{pmatrix} \mathrm{tr}[\mathbf{A}^{(n)}_1 - \mathbf{B}^{(n)}_1] \\ \vdots \\ \mathrm{tr}[\mathbf{A}^{(n)}_H - \mathbf{B}^{(n)}_H] \end{pmatrix}$$

are asymptotically normal with mean zero and covariance matrix $\frac{1}{k}\mathbf{I}_H$, *whereas*

$$\hat{\mathbf{R}}^{(n)\mathrm{f}}_{H,\boldsymbol{\theta}} - \frac{\zeta_{k,f}}{k}\,[\mathbf{I}_H \otimes (\mathbf{V}^{1/2}\otimes\mathbf{V}^{-1/2})]\begin{pmatrix} \mathrm{vec}\,(\mathbf{A}^{(n)}_1 - \mathbf{B}^{(n)}_1) \\ \vdots \\ \mathrm{vec}\,(\mathbf{A}^{(n)}_H - \mathbf{B}^{(n)}_H) \end{pmatrix}$$

is, still under $\{\mathrm{P}^{(n)}_{\boldsymbol{\vartheta}+n^{-1/2}\boldsymbol{\tau}^{(n)},\mathbf{V},f}\}$, *asymptotically normal with mean zero and covariance matrix* $\frac{1}{k^2}\mathbf{I}_{k^2H}$ *(throughout, we let* $\mathbf{A}^{(n)}_r := \mathbf{0} =: \mathbf{B}^{(n)}_s$ *for any* $r > p$ *and any* $s > q$*);*

(ii) under $\{\mathrm{P}^{(n)}_{\boldsymbol{\vartheta}+n^{-1/2}\boldsymbol{\tau}^{(n)},\mathbf{V},f}\}$*,* $\hat{Q}^{(n)\mathrm{M}}_{H,\boldsymbol{\theta}}$*,* $\tilde{Q}^{(n)\mathrm{M}}_{H,\boldsymbol{\theta}}$*,* $\hat{Q}^{(n)\mathrm{M}}_{H}$*, and* $\tilde{Q}^{(n)\mathrm{M}}_{H}$ *(resp.,* $\hat{Q}^{(n)\mathrm{f}}_{H,\boldsymbol{\theta}}$*,* $\tilde{Q}^{(n)\mathrm{f}}_{H,\boldsymbol{\theta}}$*,* $\hat{Q}^{(n)\mathrm{f}}_{H}$*, and* $\tilde{Q}^{(n)\mathrm{f}}_{H}$*) are asymptotically noncentral chi-square with* H *(resp.,* k^2H*) degrees of freedom and with noncentrality parameter*

$$\frac{\zeta^2_{k,f}}{k}\sum_{h=1}^{H}(\mathrm{tr}[\mathbf{A}_h-\mathbf{B}_h])^2, \qquad \left(\textit{resp.,}\ \zeta^2_{k,f}\sum_{h=1}^{H}\|\mathbf{V}^{-1/2}(\mathbf{A}_h-\mathbf{B}_h)\mathbf{V}^{1/2}\|^2_{\mathrm{Fr}}\right), \tag{5.1}$$

where we wrote $\mathbf{A}_h := \lim_{n\to\infty}\mathbf{A}^{(n)}_h$ *and* $\mathbf{B}_h := \lim_{n\to\infty}\mathbf{B}^{(n)}_h$ *(assuming, of course, that these limits exist).*

This result deserves comments (a) on the dependence of asymptotic local powers on the innovation density f, and (b) on local consistency issues, which, as we will see, provide the most fundamental differences between full-rank and Marden runs tests.

(a) Quite remarkably, local powers admit non-trivial lower bounds with respect to the innovation density f, which will lead to lower bounds for the AREs derived in the next section. Integrating by parts, then applying Jensen's inequality (with respect to the measure $(\mu_{k-1,f})^{-1}r^{k-1}f(r)$ and with convex function $g(x)=1/x$), we obtain that $\xi_{k,f}=(k-1)\mu_{k-2,f}/\mu_{k-1,f}\geq(k-1)\mu_{k-1,f}/\mu_{k,f}$, where the inequality is sharp (the equality is achieved at boundary points of $\mathcal{F}_{\mathrm{ULAN}}$ corresponding to elliptical distributions that are supported in the boundary of an hyperpellipsoid in $\mathbb{R}^k$). Hence, we have that

$$\inf_{f\in\mathcal{F}_{\mathrm{ULAN}}}\zeta_{k,f}=\frac{k-1}{k}, \tag{5.2}$$

where the infimum is not reached in $\mathcal{F}_{\mathrm{ULAN}}$. Also, explicit expressions for this functional coefficient $\zeta_{k,f}$ involving the regular Gamma function can be obtained at Student, power-exponential, and Gaussian densities; more precisely, we have that

$$\zeta_{k,f}=\frac{4\,\Gamma^2(\frac{k+1}{2})\Gamma^2(\frac{\nu+1}{2})}{k\,(\nu-1)\Gamma^2(\frac{k}{2})\Gamma^2(\frac{\nu}{2})} \quad\text{and}\quad \zeta_{k,f}=\frac{2\eta\Gamma(\frac{k+2\eta-1}{2\eta})\Gamma(\frac{k+1}{2\eta})}{k\Gamma^2(\frac{k}{2\eta})}, \tag{5.3}$$

at Student densities with $\nu(>2)$ degrees of freedom and at power-exponential densities with parameter η, respectively; see Section 2. The value at Gaussian densities, namely $\zeta_{k,f} = 2\,\Gamma^2(\frac{k+1}{2})/(k\Gamma^2(\frac{k}{2}))$, can be obtained by taking the limit as $\nu \to \infty$ in the first equality of (5.3), or, equivalently, by evaluating the second one at $\eta = 1$.

(b) Note that the lower bound just derived in (5.2) implies that any lack of consistency of the proposed runs tests may only come from the VARMA coefficients $\mathbf{A}_h$ and $\mathbf{B}_h$ (even for $k = 1$, since no $f \in \mathcal{F}_{\mathrm{ULAN}}$ achieves the infimum in (5.2)). If $\mathbf{A}_h = \mathbf{B}_h$ for all h, then the noncentrality parameters in (5.1) vanish, so that the nonnull asymptotic distribution of our multivariate runs test statistics is simply (central) chi-square, which in turn implies that the asymptotic powers then coincide with the nominal level α. This was, however, totally expected since such local "alternatives", by cancelling roots of the corresponding vector AR and vector MA polynomials, cannot be stochastically distinguished from the null hypothesis of randomness. What is more problematic, but also partly characterizes the Marden runs tests, is the fact that these tests cannot detect non-trivial VARMA alternatives with zero trace matrix coefficients: *Marden runs tests actually are tests for* $\mathcal{H}_0^{(n)\mathrm{trace}} : \mathrm{tr}[\mathbf{A}_r] = 0 = \mathrm{tr}[\mathbf{B}_s]$ *for all* r, s, *whereas full-rank runs tests*—which do not suffer such local consistency problems—*are genuine tests for* $\mathcal{H}_0^{(n)\mathrm{full}} : \mathbf{A}_r = \mathbf{0} = \mathbf{B}_s$ *for all* r, s. This can be justified by showing that the Marden runs tests (resp., full-rank runs tests) are nothing but the sign versions of the locally and asymptotically optimal (in the Le Cam sense) tests for $\mathcal{H}_0^{(n)\mathrm{trace}}$ (resp., for $\mathcal{H}_0^{(n)\mathrm{full}}$); see Paindaveine (2009) for details.

With this in mind, we can come back to the real data example of Section 4, and comment on the fact that, for the series of squared log returns, $\hat{\phi}_1^{(n)\mathrm{f}}$ rejects the null of randomness $\mathcal{H}^{(n)}$ while $\hat{\phi}_1^{(n)\mathrm{M}}$ does not: since it is quite likely that any couple of marginal first-order autocorrelations $\rho_\ell(1)$, $\ell = 1, 2, 3, 4$ with large absolute values (if any) would share the same sign, we essentially learn that each $\rho_\ell(1)$ is probably small (from the fact that $\hat{\phi}_1^{(n)\mathrm{M}}$ does not reject the null), and that the first-order cross-autocorrelations $\rho_{\ell,\ell'}(1)$, $\ell, \ell' = 1, 2, 3, 4$, $\ell \neq \ell'$, are large enough to cause rejection of the null by $\hat{\phi}_1^{(n)\mathrm{f}}$. As announced in Section 4, this suggests that no index strongly determine their own future volatility while there are some indices whose volatility is mainly driven by other indices. Although such a highly non-robust estimator should be considered with caution, the VAR(1) (with a constant) OLS estimator of $\mathbf{A}_1$, which, for the series of squared log returns considered, yields

$$\hat{\mathbf{A}}_1 = \begin{pmatrix} .02 & .03 & .29 & .00 \\ -.02 & -.04 & -.43 & -.01 \\ -.01 & -.01 & -.04 & .01 \\ .04 & .01 & .00 & .04 \end{pmatrix}$$

(hence, $(\mathrm{tr}[\hat{\mathbf{A}}_1])^2 = 2e{-}04$ and $\|\hat{\mathbf{A}}_1\|^2_{\mathrm{Fr}} = .28$), essentially supports these claims.

5.3 Asymptotic relative efficiencies.

The natural benchmarks for efficiency comparisons are Gaussian Portmanteau-type tests that reject the null of randomness $\mathcal{H}_{\boldsymbol{\theta}}^{(n)}$ when the standard (in this $\boldsymbol{\theta}$-specified setup, $\boldsymbol{\theta}$-centered) correlation matrices $\boldsymbol{\rho}_{h,\boldsymbol{\theta}}^{(n)\mathcal{N}} := (\boldsymbol{\gamma}_{0,\boldsymbol{\theta}}^{(n)\mathcal{N}})^{-1/2}\boldsymbol{\gamma}_{h,\boldsymbol{\theta}}^{(n)\mathcal{N}}(\boldsymbol{\gamma}_{0,\boldsymbol{\theta}}^{(n)\mathcal{N}})^{-1/2}$, $h = 1,\ldots,H$, with $\boldsymbol{\gamma}_{h,\boldsymbol{\theta}}^{(n)\mathcal{N}} := \frac{1}{n-h}\sum_{t=h+1}^{n}(\mathbf{X}_t-\boldsymbol{\theta})(\mathbf{X}_{t-h}-\boldsymbol{\theta})'$, are too large (in some adequate norm). Denote by $\mathcal{H}_{\boldsymbol{\theta},\boldsymbol{\Sigma}}^{(n)2}$ the hypothesis under which the observations $\mathbf{X}_1,\ldots,\mathbf{X}_n$ are i.i.d. with mean $\boldsymbol{\theta}$ and full-rank covariance matrix $\boldsymbol{\Sigma}$ (the superscript 2 in the notation stresses this second-order moment assumption). Under $\mathcal{H}_{\boldsymbol{\theta},\boldsymbol{\Sigma}}^{(n)2}$, we have that $\mathbf{T}_{H,\boldsymbol{\theta}}^{(n)\mathcal{N}} := ((n-1)^{1/2}(\mathrm{vec}\,\boldsymbol{\rho}_{1,\boldsymbol{\theta}}^{(n)\mathcal{N}})',\ldots,(n-H)^{1/2}(\mathrm{vec}\,\boldsymbol{\rho}_{H,\boldsymbol{\theta}}^{(n)\mathcal{N}})')' = [\mathbf{I}_H\otimes(\boldsymbol{\Sigma}\otimes\boldsymbol{\Sigma})]^{-1/2}((n-1)^{1/2}(\mathrm{vec}\,\boldsymbol{\gamma}_{1,\boldsymbol{\theta}}^{(n)\mathcal{N}})',\ldots,(n-H)^{1/2}(\mathrm{vec}\,\boldsymbol{\gamma}_{H,\boldsymbol{\theta}}^{(n)\mathcal{N}})')' + o_{\mathrm{P}}(1)$ (as $n\to\infty$) is asymptotically standard normal. Hence, the resulting Portmanteau test—$\phi_{H,\boldsymbol{\theta}}^{(n)\mathcal{N}}$, say—rejects the null of randomness $\mathcal{H}_{\boldsymbol{\theta}}^{(n)2} := \cup_{\boldsymbol{\Sigma}}\mathcal{H}_{\boldsymbol{\theta},\boldsymbol{\Sigma}}^{(n)2}$ at asymptotic level α whenever

$$Q_{H,\boldsymbol{\theta}}^{(n)\mathcal{N}} := \|\mathbf{T}_{H,\boldsymbol{\theta}}^{(n)\mathcal{N}}\|^2 = \sum_{h=1}^{H}(n-h)\|\boldsymbol{\rho}_{h,\boldsymbol{\theta}}^{(n)\mathcal{N}}\|^2_{\mathrm{Fr}} > \chi^2_{k^2H,1-\alpha}. \tag{5.4}$$

Unlike the runs tests of Sections 3, $\phi_{H,\boldsymbol{\theta}}^{(n)\mathcal{N}}$ requires second-order moment assumptions and cannot deal with heteroskedasticity. It is also poorly resistant to possible outliers (see Section 6). Its only advantage over the proposed runs tests is actually that it does not require the underlying distribution to have elliptical directions.

Turning to power considerations (and restricting to the VARMA alternatives of Section 5.2), it can be shown that, provided that the observations are i.i.d. with an elliptical distribution admitting finite second-order moments, $\phi_{H,\boldsymbol{\theta}}^{(n)\mathcal{N}}$ is asymptotically equivalent to the locally and asymptotically optimal Gaussian test of randomness introduced in Hallin and Paindaveine (2002b, Section 2.3). The Portmanteau test $\phi_{H,\boldsymbol{\theta}}^{(n)\mathcal{N}}$ is therefore optimal under Gaussian innovations. More generally, asymptotic local powers can be obtained along the same lines as in Proposition 5.3.

Proposition 5.4 *Fix $\boldsymbol{\vartheta} = (\boldsymbol{\theta}', \mathbf{0}')' \in \mathbb{R}^L$, $\mathbf{V} \in \mathcal{M}_k$, $f \in \mathcal{F}_{\mathrm{ULAN}}$, and $H \in \mathbb{N}_0$. Then, under $\mathrm{P}^{(n)}_{\boldsymbol{\vartheta}+n^{-1/2}\boldsymbol{\tau}^{(n)},\mathbf{V},f}$, $Q_{H,\boldsymbol{\theta}}^{(n)\mathcal{N}}$ is asymptotically noncentral chi-square with k^2H degrees of freedom and with noncentrality parameter $\sum_{h=1}^{H} \|\mathbf{V}^{-1/2}(\mathbf{A}_h - \mathbf{B}_h)\mathbf{V}^{1/2}\|^2_{\mathrm{Fr}}$, where $\mathbf{A}_h$ and $\mathbf{B}_h$ are defined as in (5.1).*

These tests can also be turned into tests for $\mathcal{H}^{(n)}$ by replacing $\boldsymbol{\theta}$ with any root-n consistent estimate (the sample average is a natural candidate for such a Gaussian procedure, though). Similarly as for the runs tests above, the asymptotic properties of the resulting Gaussian Portmanteau tests of randomness ($\phi_H^{(n)\mathcal{N}}$, based on $Q_H^{(n)\mathcal{N}}$ say) are not affected by this replacement. This is actually a corollary of the next result (which easily follows from the continuous mapping theorem).

Lemma 5.2 *Fix $\boldsymbol{\theta} \in \mathbb{R}^k$ and $h \in \mathbb{N}$, and let $\hat{\boldsymbol{\theta}}$ be a root-n consistent estimate of $\boldsymbol{\theta}$ under $\mathcal{H}_{\boldsymbol{\theta}}^{(n)2} = \cup_{\boldsymbol{\Sigma}} \mathcal{H}_{\boldsymbol{\theta},\boldsymbol{\Sigma}}^{(n)2}$. Then, under $\mathcal{H}_{\boldsymbol{\theta}}^{(n)2}$, $\boldsymbol{\gamma}_{h,\hat{\boldsymbol{\theta}}}^{(n)\mathcal{N}} - \boldsymbol{\gamma}_{h,\boldsymbol{\theta}}^{(n)\mathcal{N}}$ is $o_{\mathrm{P}}(n^{-1/2})$ as $n \to \infty$.*

Propositions 5.3 and 5.4 allow to compare, by means of asymptotic relative efficiencies (AREs), the performances of the Marden and full-rank runs tests with those of the Gaussian Portmanteau tests introduced above. For full-rank runs tests, these

AREs are simply obtained by computing the ratios of the noncentrality parameters in the asymptotic nonnull distributions of $\hat{Q}^{(n)\mathrm{f}}_{H,\boldsymbol{\theta}}/\tilde{Q}^{(n)\mathrm{f}}_{H,\boldsymbol{\theta}}$ and $Q^{(n)\mathcal{N}}_{H,\boldsymbol{\theta}}$ (or $\hat{Q}^{(n)\mathrm{f}}_{H}/\tilde{Q}^{(n)\mathrm{f}}_{H}$ and $Q^{(n)\mathcal{N}}_{H}$, in the $\boldsymbol{\theta}$-unspecified case). For Marden runs tests, however, the degrees of freedom in the limiting distributions of $\hat{Q}^{(n)\mathrm{M}}_{H,\boldsymbol{\theta}}/\tilde{Q}^{(n)\mathrm{M}}_{H,\boldsymbol{\theta}}$ and $Q^{(n)\mathcal{N}}_{H,\boldsymbol{\theta}}$ do not match in the multivariate case ($k \geq 2$), and a direct use of the ratio of noncentrality parameters is therefore no longer valid. We then use the extension of the concept of Pitman ARE to cases where the limiting distributions of the competing tests are of different types; see Nyblom and Mäkeläinen (1983). In such cases, the resulting relative efficiency may depend on the asymptotic level α and power β. The general result is the following.

Proposition 5.5 *Fix $\boldsymbol{\vartheta} = (\boldsymbol{\theta}', \mathbf{0}')' \in \mathbb{R}^L$, $\mathbf{V} \in \mathcal{M}_k$, $f \in \mathcal{F}_{\mathrm{ULAN}}$, and $H \in \mathbb{N}_0$, and let Assumption ($A_{\boldsymbol{\theta}}$) hold. Let $c^{\alpha,\beta}_{\ell}$ be such that the upper-β quantile of the noncentral χ^2_ℓ distribution with noncentrality parameter $c^{\alpha,\beta}_{\ell}$ is $\chi^2_{\ell,1-\alpha}$. Then, when testing $\mathrm{P}^{(n)}_{\boldsymbol{\vartheta},\mathbf{V},f}$ against $\mathrm{P}^{(n)}_{\boldsymbol{\vartheta}+n^{-1/2}\boldsymbol{\tau}^{(n)},\mathbf{V},f}$, the ARE of $\hat{\phi}^{(n)\mathrm{M}}_{H,\boldsymbol{\theta}}/\tilde{\phi}^{(n)\mathrm{M}}_{H,\boldsymbol{\theta}}$ (resp., of $\hat{\phi}^{(n)\mathrm{f}}_{H,\boldsymbol{\theta}}/\tilde{\phi}^{(n)\mathrm{f}}_{H,\boldsymbol{\theta}}$) with respect to $\phi^{(n)\mathcal{N}}_{H,\boldsymbol{\theta}}$ coincide with the ARE of $\hat{\phi}^{(n)\mathrm{M}}_{H}/\tilde{\phi}^{(n)\mathrm{M}}_{H}$ (resp., $\hat{\phi}^{(n)\mathrm{f}}_{H}/\tilde{\phi}^{(n)\mathrm{f}}_{H}$) with respect to $\phi^{(n)\mathcal{N}}_{H}$, and are given by*

$$\frac{c^{\alpha,\beta}_{k^2H}}{c^{\alpha,\beta}_{H}} \times \zeta^2_{k,f} \times \frac{\frac{1}{k}\sum_{h=1}^{H}(\mathrm{tr}[\mathbf{A}_h - \mathbf{B}_h])^2}{\sum_{h=1}^{H}\|\mathbf{V}^{-1/2}(\mathbf{A}_h - \mathbf{B}_h)\mathbf{V}^{1/2}\|^2_{\mathrm{Fr}}} \qquad \Big(\textit{resp.},\ \ \zeta^2_{k,f}\Big), \tag{5.5}$$

where $\mathbf{A}_h$ and $\mathbf{B}_h$ are defined as in (5.1) and where α (resp., β) is the common asymptotic size (resp., power) of the tests.

The three factors of the first ARE in (5.5) are of a very different nature. The "degrees of freedom" factor $c^{\alpha,\beta}_{k^2H}/c^{\alpha,\beta}_{H}$ takes into account the fact the Marden runs tests start with some advantage ($c^{\alpha,\beta}_{k^2H}/c^{\alpha,\beta}_{H} \geq 1$) over their Gaussian Portmanteau competitors since they are based on a smaller number of degrees of freedom. The "functional" factor $\zeta^2_{k,f}$ is the only one depending on the underlying innovation density f, and, roughly speaking, measures the relative performance—at f—of sign procedures with respect to their parametric Gaussian counterparts (this factor is indeed the only one in the second ARE in (5.5)). Eventually, the "structural" factor $\frac{1}{k}\sum_{h=1}^{H}(\mathrm{tr}[\mathbf{A}_h - \mathbf{B}_h])^2 / \sum_{h=1}^{H} \|\mathbf{V}^{-1/2}(\mathbf{A}_h - \mathbf{B}_h)\mathbf{V}^{1/2}\|^2_{\mathrm{Fr}}$ only depends on the type of VARMA alternative considered and on the underlying shape $\mathbf{V}$. It is always equal to one for univariate series ($k = 1$), but crucially affects the first ARE values in (5.5) in the multivariate case.

Table 2 provides some numerical values of the functional factor $\zeta^2_{k,f}$ for various underlying densities, namely for several t, Gaussian, and power-exponential densities, as well as for some boundary cases. Table 2 illustrates that, as expected from (5.2), the asymptotic efficiency of multivariate runs tests may be relatively poor for very small dimensions k (particularly so in the univariate case, to which runs tests have been limited so far), but that it becomes uniformly good as k increases (it directly follows from (5.2) that the AREs of full-rank runs tests with respect to Gaussian Portmanteau tests are always larger than $(\frac{k-1}{k})^2$ in dimension k). This is in line with the "sufficiency-type" properties that sign statistics enjoy for large k; see typically Hall et al. (2005). Irrespective of k, runs tests perform best under heavy tails—as it is often the case for sign procedures.

6 A MONTE-CARLO STUDY.

Several simulations were conducted in order to investigate the finite-sample behavior of the proposed multivariate runs tests for $k = 2$. Starting throughout with i.i.d. (either bivariate standard normal or bivariate standard t_3) random vectors $\boldsymbol{\eta}_t$, samples $(\mathbf{X}_1, \ldots, \mathbf{X}_n)$ of size $n = 200$ were generated as the last n random vectors in the series $(\mathbf{Y}_1, \ldots, \mathbf{Y}_{500+n})$ obtained from the VAR "alternatives"

$$(\mathbf{Y}_t - \boldsymbol{\theta}) - \left(\tfrac{m}{M}\mathbf{A}_1^{(\ell)}\right)(\mathbf{Y}_{t-1} - \boldsymbol{\theta}) - \left(\tfrac{m}{M}\mathbf{A}_2^{(\ell)}\right)(\mathbf{Y}_{t-2} - \boldsymbol{\theta}) = \boldsymbol{\varepsilon}_t \ (:= \mathbf{V}^{1/2}\boldsymbol{\eta}_t), \quad t \geq 3,$$

$$\mathbf{Y}_1 = \mathbf{Y}_2 = \boldsymbol{\theta} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \mathbf{V} = \begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix}, \quad m = 0, 1, 2, \ldots, 6 = M,$$

with Type 1 or Type 2 alternatives according to

$$\mathbf{A}_1^{(1)} = \mathbf{A}_2^{(1)} = \begin{pmatrix} .15 & 0 \\ 0 & .15 \end{pmatrix} \quad \text{or} \quad \mathbf{A}_1^{(2)} = \mathbf{A}_2^{(2)} = \begin{pmatrix} .05 & .10 \\ .10 & -.05 \end{pmatrix}, \qquad (6.1)$$

respectively. Contaminated data sets were also obtained by replacing $\mathbf{X}_t$, $t = 10, 12, 14$, with $10\mathbf{X}_t + \mathbf{b}$, where $\mathbf{b} = (3, -7)'$ (we avoided $\mathbf{b} = (0, 0)'$ since our runs tests, at least in their $\boldsymbol{\theta} = (0, 0)'$-specified versions, would totally be insensitive to such *radial* outliers). Each of the three simulations (A)-(C) below is based on $N = 10{,}000$ replications, and all tests were performed at asymptotic level $\alpha = 5\%$.

(A) In a first simulation, we considered the $\boldsymbol{\theta}$-specified tests $\hat{\phi}^{(n)\mathrm{M}}_{2,\boldsymbol{\theta}}$, $\hat{\phi}^{(n)\mathrm{f}}_{2,\boldsymbol{\theta}}$, and $\phi^{(n)\mathcal{N}}_{2,\boldsymbol{\theta}}$ under Gaussian and t_3 innovations, and for both types of VAR alternatives above. The shape estimator $\hat{\mathbf{V}}$ used in both runs tests is Tyler's estimator $\hat{\mathbf{V}}_{\mathrm{Tyl}}$ in (3.9). The resulting power curves are presented in Figure 3. All tests seem to meet the level constraint. Under Gaussian and t_3 innovations, the AREs of $\hat{\phi}^{(n)\mathrm{f}}_{2,\boldsymbol{\theta}}$ with respect to $\phi^{(n)\mathcal{N}}_{2,\boldsymbol{\theta}}$ are equal to .617 and 1.000, respectively (for both types of alternatives considered), which is perfectly reflected in Figure 3. As expected, Marden runs test collapse under Type 2 alternatives. Under Type 1 alternatives, they behave in accordance with Proposition 5.5, which allows to show that the AREs of $\hat{\phi}^{(n)\mathrm{M}}_{2,\boldsymbol{\theta}}$ with respect to $\phi^{(n)\mathcal{N}}_{2,\boldsymbol{\theta}}$, under such alternatives and at asymptotic level $\alpha = .5$/power $\beta = .80$, are equal to .962 and 1.559, at Gaussian and t_3 innovations, respectively. As expected from (5.1), powers are higher under bivariate t_3 innovations (for which $\zeta^2_{k,f} = 1.000$) than under bivariate Gaussian ones (for which $\zeta^2_{k,f} = .617$).

(B) The second simulation restricts to Type 1 alternatives with t_3 innovations (but considers both clean and contaminated data), and compares the $\boldsymbol{\theta}$-specified tests $\hat{\phi}^{(n)\mathrm{M}}_{2,\boldsymbol{\theta}}$, $\hat{\phi}^{(n)\mathrm{f}}_{2,\boldsymbol{\theta}}$, and $\phi^{(n)\mathcal{N}}_{2,\boldsymbol{\theta}}$ involved in the first simulation with their $\boldsymbol{\theta}$-unspecified counterparts $\hat{\phi}^{(n)\mathrm{M}}_{2}$, $\hat{\phi}^{(n)\mathrm{f}}_{2}$, and $\phi^{(n)\mathcal{N}}_{2}$. The tests $\hat{\phi}^{(n)\mathrm{M}}_{2,\boldsymbol{\theta}}/\hat{\phi}^{(n)\mathrm{f}}_{2,\boldsymbol{\theta}}$ (resp., $\hat{\phi}^{(n)\mathrm{M}}_{2}/\hat{\phi}^{(n)\mathrm{f}}_{2}$) are based on $\hat{\mathbf{V}}_{\mathrm{Tyl}}$ (resp., $(\hat{\boldsymbol{\theta}}_{\mathrm{HR}}, \hat{\mathbf{V}}_{\mathrm{HR}})$), whereas $\phi^{(n)\mathcal{N}}_{2}$ uses $\hat{\boldsymbol{\theta}} = \bar{\mathbf{X}}$, which is the natural location estimate for a parametric Gaussian procedure. The results (see Figure 4) show that $\boldsymbol{\theta}$-unspecified tests are, at $n = 200$, slightly less powerful than their $\boldsymbol{\theta}$-specified counterparts. For clean data, the hierarchy of the various types of tests is of course the same as in the first simulation (with Type 1 alternatives). For contaminated data, however, Gaussian tests become useless (their size under the null is close to 70%), whereas runs tests barely seem to be affected by this contamination.

(C) Finally, a last simulation compares the $\boldsymbol{\theta}$-specified shape-based tests $\hat{\phi}^{(n)\mathrm{M}}_{2,\boldsymbol{\theta}}/\hat{\phi}^{(n)\mathrm{f}}_{2,\boldsymbol{\theta}}$ with their hyperplane-based counterparts $\tilde{\phi}^{(n)\mathrm{M}}_{2,\boldsymbol{\theta}}/\tilde{\phi}^{(n)\mathrm{f}}_{2,\boldsymbol{\theta}}$. Again, only Type 1 alternatives with t_3 innovations are considered. Preliminary simulations, with $n = 200$, showed that shape-based tests and hyperplane-based tests behave in exactly the same way (at least for clean data sets). Hence, we rather present results for $n = 20$ (see Figure 5), with a common value of $\mathbf{A}^{(1)}_1$ and $\mathbf{A}^{(1)}_2$ multiplied by 3 ($\approx \frac{\sqrt{200}}{\sqrt{20}}$) when compared to (6.1), in order to obtain comparable powers. At this small sample size, we learn that, for clean data sets, shape-based tests slightly dominate the hyperplane-based ones, while the latter seem to be better resistant to contamination than the former.

As a conclusion, these simulations nicely confirm the asymptotic results of the previous sections, but also suggest that hyperplane-based runs tests are (even) more robust than their shape-based counterparts, hence should be favored for small samples. However, for large sample size n and/or moderate-to-large dimension k, hyperplane-based tests, unlike shape-based ones, are computationally intractable; see the real data example. Finally, we stress that these simulations were actually conducted in the least favourable case for our multivariate runs tests, namely the bivariate case:

remember indeed that efficiency of runs tests increases with the dimension k. Yet, even for $k = 2$, runs tests appear as nicely efficient procedures when compared to their classical Portmanteau competitors.

7 FINAL COMMENTS.

The runs tests of randomness developed in this work maintain in the multivariate setup the properties that made the success of their univariate classical versions: high flexibility, very good robustness, validity under heteroskedastic patterns, etc. Better than that: the main drawback of univariate runs tests, namely their poor efficiency under light tails, vanishes as the dimension k increases. For moderate-to-large k, runs tests are therefore procedures that combine efficiency and robustness (and ease of computation, if one restricts to shape-based runs tests).

The original motivation of this work was to evaluate procedures based on Marden's multivariate runs in (1.3). One might think that the conclusion of this paper is that Marden-type runs tests should be avoided because of their severe lack of consistency. Although we indeed want to warn practitioners about these consistency issues, we point out that Marden runs tests, when used in combination with the full-rank runs tests based on (1.4), may help obtaining information on the reason why rejection occurs or not; see Section 4. Hence, despite our preference for procedures based on full-rank runs, our recommendation would be to use both types of runs tests together.

Eventually, we point out that full-rank runs, just as Marden-type runs, might also be useful in other contexts. For instance, Marden (1999) proposed to define runs tests of spherical symmetry by basing on (1.3) appropriate extensions of the McWilliams (1990) runs test of univariate symmetry. Clearly, full-rank runs tests of spherical (or elliptical) symmetry could be defined by using the methodology proposed in the present paper. This, as well as the development of multiple-output two-sample location tests based on our multivariate runs, will be the topic of a future work.

A APPENDIX: PROOFS.

Proof of Lemma 3.1. This L^2-convergence result follows by applying Lebesgue's DCT in Lemma 1 of Hallin and Paindaveine (2002a) (note that the $\mathbf{X}_t$'s there do not need be i.i.d. spherical random vectors, but only that their projections $\mathbf{X}_t/\|\mathbf{X}_t\|$ on the unit sphere $\mathcal{S}^{k-1}$ are i.i.d. uniformly distributed on $\mathcal{S}^{k-1}$). $\square$

Proof of Lemma 5.1. (i) We start with the proof that $\mathrm{E}[\|\hat{\mathbf{R}}^{(n)\mathrm{f}}_{H,\boldsymbol{\theta}} - \mathbf{R}^{(n)\mathrm{f}}_{H,\boldsymbol{\theta};\mathbf{V}}\|^2]$ is $o(1)$ as $n \to \infty$. Write $\mathbf{U}_t$, $\hat{\mathbf{U}}_t$, and $\hat{\mathbf{W}}_{t,t-h}$ for $\mathbf{U}_{t,\boldsymbol{\theta}}(\mathbf{V})$, $\mathbf{U}_{t,\boldsymbol{\theta}}(\hat{\mathbf{V}})$, and $\mathbf{U}_{t,\boldsymbol{\theta}}(\hat{\mathbf{V}})\mathbf{U}'_{t-h,\boldsymbol{\theta}}(\hat{\mathbf{V}})$, respectively. Letting $\mathbf{Y}_t := \mathrm{Sign}[(\mathbf{X}_t - \boldsymbol{\theta})_1](\mathbf{X}_t - \boldsymbol{\theta})$, Assumption $(\mathrm{A}_{\boldsymbol{\theta}})$(ii) then yields

$$\mathrm{E}[(\hat{\mathbf{W}}_{s,s-h} - \mathbf{U}_s\mathbf{U}'_{s-h})(\hat{\mathbf{W}}_{t,t-h} - \mathbf{U}_t\mathbf{U}'_{t-h}) \,|\, \mathbf{X}_1, \ldots, \mathbf{X}_{t-1}, \mathbf{Y}_t, \mathbf{X}_{t+1}, \ldots, \mathbf{X}_n] = 0, \quad \text{(A.1)}$$

for any $h+1 \leq s < t \leq n$. Writing $\hat{U}_{t;i}$ and $U_{t;i}$ for the ith component of $\hat{\mathbf{U}}_t$ and $\mathbf{U}_t$ respectively, (A.1) and Assumption $(\mathrm{A}_{\boldsymbol{\theta}})$(iii) yield that, for any $i, j = 1, \ldots, k$, $\mathrm{E}[((\hat{\mathbf{r}}^{(n)\mathrm{f}}_{h,\boldsymbol{\theta}} - \mathbf{r}^{(n)\mathrm{f}}_{h,\boldsymbol{\theta};\mathbf{V}})_{ij})^2] = (n-h)^{-1}\sum_{t=h+1}^n \mathrm{E}[(\hat{U}_{t;i}\hat{U}_{t-h;j} - U_{t;i}U_{t-h;j})^2] = \mathrm{E}[(\hat{U}_{h+1;i}\hat{U}_{1;j} - U_{h+1;i}U_{1;j})^2] \leq 2\,\mathrm{E}[(\hat{U}_{h+1;i} - U_{h+1;i})^2\hat{U}^2_{1;j}] + 2\,\mathrm{E}[U^2_{h+1;i}(\hat{U}_{1;j} - U_{1;j})^2] \leq 4\,\mathrm{E}[\|\hat{\mathbf{U}}_1 - \mathbf{U}_1\|^2]$. Now, with $\|\mathbf{C}\|_{\mathcal{L}} := \sup\{\|\mathbf{C}\mathbf{x}\| \,:\, \|\mathbf{x}\| = 1\}$, we have $\|\hat{\mathbf{U}}_1 - \mathbf{U}_1\| \leq 2\,\|\hat{\mathbf{V}}^{-1/2}(\mathbf{X}_1 - \boldsymbol{\theta}) - \mathbf{V}^{-1/2}(\mathbf{X}_1 - \boldsymbol{\theta})\|/\|\mathbf{V}^{-1/2}(\mathbf{X}_1 - \boldsymbol{\theta})\| \leq 2\,\|\hat{\mathbf{V}}^{-1/2} - \mathbf{V}^{-1/2}\|_{\mathcal{L}}\,\|\mathbf{V}^{1/2}\|_{\mathcal{L}}$, which, by Assumption $(\mathrm{A}_{\boldsymbol{\theta}})$(i), is $o(1)$ as $n \to \infty$ in probability (hence, also in quadratic mean, since $\|\hat{\mathbf{U}}_1 - \mathbf{U}_1\|$ is bounded). We conclude that $\mathrm{E}[\|\hat{\mathbf{R}}^{(n)\mathrm{f}}_{H,\boldsymbol{\theta}} - \mathbf{R}^{(n)\mathrm{f}}_{H,\boldsymbol{\theta};\mathbf{V}}\|^2] = \sum_{h=1}^H \sum_{i,j=1}^k \mathrm{E}[((\hat{\mathbf{r}}^{(n)\mathrm{f}}_{h,\boldsymbol{\theta}} - \mathbf{r}^{(n)\mathrm{f}}_{h,\boldsymbol{\theta};\mathbf{V}})_{ij})^2]$ is indeed $o(1)$ as $n \to \infty$.

Since $\hat{\mathbf{R}}^{(n)\mathrm{M}}_{H,\boldsymbol{\theta}} - \mathbf{R}^{(n)\mathrm{M}}_{H,\boldsymbol{\theta};\mathbf{V}} = [\mathbf{I}_H \otimes (\mathrm{vec}\,\mathbf{I}_k)'](\hat{\mathbf{R}}^{(n)\mathrm{f}}_{H,\boldsymbol{\theta}} - \mathbf{R}^{(n)\mathrm{f}}_{H,\boldsymbol{\theta};\mathbf{V}})$, the result for $\hat{\mathbf{R}}^{(n)\mathrm{M}}_{H,\boldsymbol{\theta}}$ directly follows from the one on $\hat{\mathbf{R}}^{(n)\mathrm{f}}_{H,\boldsymbol{\theta}}$. As for $\tilde{\mathbf{R}}^{(n)\mathrm{M}}_{H,\boldsymbol{\theta}}$, the "hyperplane-based" analog of (A.1) states that, for $h+1 \leq s < t \leq n$, $\mathrm{E}[(\cos(\pi p^{(n)}_{s,s-h,\boldsymbol{\theta}}) - \mathbf{U}'_{s,\boldsymbol{\theta}}\mathbf{U}_{s-h,\boldsymbol{\theta}})(\cos(\pi p^{(n)}_{t,t-h,\boldsymbol{\theta}}) - \mathbf{U}'_{t,\boldsymbol{\theta}}\mathbf{U}_{t-h,\boldsymbol{\theta}}) \,|\, \mathbf{X}_1, \ldots, \mathbf{X}_{t-1}, \mathbf{Y}_t, \mathbf{X}_{t+1}, \ldots, \mathbf{X}_n] = 0$ (note that $\cos(\pi p^{(n)}_{s,t,\boldsymbol{\theta}})$ only changes sign if $\mathbf{X}_t$ is reflected with respect to $\boldsymbol{\theta}$). Therefore, working along the same lines as above, we obtain, by using Lemma 3.1, that $\mathrm{E}[\|\tilde{\mathbf{R}}^{(n)\mathrm{M}}_{H,\boldsymbol{\theta}} - \mathbf{R}^{(n)\mathrm{M}}_{H,\boldsymbol{\theta};\mathbf{V}}\|^2] = \sum_{h=1}^{H} \mathrm{E}[(\tilde{r}^{(n)\mathrm{M}}_{h,\boldsymbol{\theta}} - r^{(n)\mathrm{M}}_{h,\boldsymbol{\theta};\mathbf{V}})^2] = \sum_{h=1}^{H} \mathrm{E}[(\cos(\pi p^{(n)}_{h+1,1,\boldsymbol{\theta}}) - \mathbf{U}'_{h+1,\boldsymbol{\theta}}\mathbf{U}_{1,\boldsymbol{\theta}})^2]$ is $o(1)$ as $n \to \infty$.

(ii) Since standard CLTs show that, under $\mathcal{H}^{(n)}_{\boldsymbol{\theta},\mathbf{V}}$, $\mathbf{R}^{(n)\mathrm{M}}_{H,\boldsymbol{\theta};\mathbf{V}}$ (resp., $\mathbf{R}^{(n)\mathrm{f}}_{H,\boldsymbol{\theta};\mathbf{V}}$) is asymptotically normal with mean zero and covariance matrix $\frac{1}{k}\mathbf{I}_H$ (resp., $\frac{1}{k^2}\mathbf{I}_{k^2H}$), Part (i) of the Lemma yields the result. $\square$

Proof of Proposition 5.1. (i) Since the statistic $\hat{\mathbf{R}}^{(n)\mathrm{f}}_H$ is affine-equivariant, we may assume that $(\boldsymbol{\theta}, \mathbf{V}) = (\mathbf{0}, \mathbf{I}_k)$. Writing then $\hat{\mathbf{U}}_t$ and $\mathbf{U}_t$ for $\mathbf{U}_{t,\hat{\boldsymbol{\theta}}_{\mathrm{HR}}}(\hat{\mathbf{V}}_{\mathrm{HR}})$ and $\mathbf{U}_{t,\mathbf{0}}(\mathbf{I}_k)$, respectively, we have that (see the proof of Theorem 1 in Taskinen et al. 2003)

$$\hat{\mathbf{U}}_t = \mathbf{U}_t + (\mathbf{U}_t\mathbf{U}'_t - \mathbf{I}_k)\Big[\|\mathbf{X}_t\|^{-1}\,\hat{\boldsymbol{\theta}}_{\mathrm{HR}} + \tfrac{1}{2}(\hat{\mathbf{V}}_{\mathrm{HR}} - \mathbf{I}_k)\mathbf{U}_t\Big] + o_{\mathrm{P}}(n^{-1/2}), \qquad \text{(A.2)}$$

as $n \to \infty$ under $\mathcal{H}^{(n)}_{\mathbf{0},\mathbf{I}_k}$. It easily follows from (A.2) that, for any positive integer h, $\hat{\mathbf{r}}^{(n)\mathrm{f}}_h - \mathbf{r}^{(n)\mathrm{f}}_{h,\mathbf{0};\mathbf{I}_k} := (n-h)^{-1/2}\sum_{t=h+1}^{n}(\hat{\mathbf{U}}_t\hat{\mathbf{U}}'_{t-h} - \mathbf{U}_t\mathbf{U}'_{t-h}) = o_{\mathrm{P}}(1)$ as $n \to \infty$, under $\mathcal{H}^{(n)}_{\mathbf{0},\mathbf{I}_k}$, which of course implies that, still under $\mathcal{H}^{(n)}_{\mathbf{0},\mathbf{I}_k}$, $\hat{\mathbf{R}}^{(n)\mathrm{f}}_H - \mathbf{R}^{(n)\mathrm{f}}_{H,\boldsymbol{\theta};\mathbf{V}} = o_{\mathrm{P}}(1)$ as $n \to \infty$. The result for $\hat{\mathbf{R}}^{(n)\mathrm{M}}_H$ then directly follows by noting that $\hat{\mathbf{R}}^{(n)\mathrm{M}}_{H,\boldsymbol{\theta}} = [\mathbf{I}_H \otimes (\mathrm{vec}\,\mathbf{I}_k)']\hat{\mathbf{R}}^{(n)\mathrm{f}}_{H,\boldsymbol{\theta}}$ and $\mathbf{R}^{(n)\mathrm{M}}_{H,\boldsymbol{\theta};\mathbf{V}} = [\mathbf{I}_H \otimes (\mathrm{vec}\,\mathbf{I}_k)']\mathbf{R}^{(n)\mathrm{f}}_{H,\boldsymbol{\theta};\mathbf{V}}$ for any k-vector $\boldsymbol{\theta}$ and any $\mathbf{V} \in \mathcal{M}_k$. Finally, the claim for $\tilde{\mathbf{R}}^{(n)\mathrm{M}}_H$ follows by working as in Randles and Um (1998).

(ii) Fix $\boldsymbol{\theta} \in \mathbb{R}^k$ and $\mathbf{V} \in \mathcal{M}_k$. Part (i) of the proposition implies that $\hat{Q}^{(n)\mathrm{M}}_H + o_{\mathrm{P}}(1) = k(\mathbf{R}^{(n)\mathrm{M}}_{H,\boldsymbol{\theta};\mathbf{V}})'\mathbf{R}^{(n)\mathrm{M}}_{H,\boldsymbol{\theta};\mathbf{V}} = \tilde{Q}^{(n)\mathrm{M}}_H + o_{\mathrm{P}}(1)$ (resp., $\hat{Q}^{(n)\mathrm{f}}_H = k^2(\mathbf{R}^{(n)\mathrm{f}}_{H,\boldsymbol{\theta};\mathbf{V}})'\mathbf{R}^{(n)\mathrm{f}}_{H,\boldsymbol{\theta};\mathbf{V}} + o_{\mathrm{P}}(1)$) as $n \to \infty$ under $\mathcal{H}^{(n)}_{\boldsymbol{\theta},\mathbf{V}}$. Hence, for $\hat{Q}^{(n)\mathrm{M}}_H$, $\tilde{Q}^{(n)\mathrm{M}}_H$, and $\hat{Q}^{(n)\mathrm{f}}_H$, the claims follow from the fact that, still under $\mathcal{H}^{(n)}_{\boldsymbol{\theta},\mathbf{V}}$, $\mathbf{R}^{(n)\mathrm{M}}_{H,\boldsymbol{\theta};\mathbf{V}}$ (resp., $\mathbf{R}^{(n)\mathrm{f}}_{H,\boldsymbol{\theta};\mathbf{V}}$) is asymptotically normal with mean zero and covariance matrix $\frac{1}{k}\mathbf{I}_H$ (resp., $\frac{1}{k^2}\mathbf{I}_{k^2H}$); see the proof of Lemma 5.1(ii). As for $\tilde{Q}^{(n)\mathrm{f}}_H$, showing that the replacement of $\boldsymbol{\theta}$ with $\hat{\boldsymbol{\theta}}_{\mathrm{HR}}$ does not affect the behavior of the test in probability can again be achieved by using the same argument as in Randles and Um (1998). $\square$

Proof of Proposition 5.2. The result follows by merging the LAN/ULAN results in Hallin and Paindaveine (2002a, 2002b, and 2005). $\square$

Proof of Proposition 5.3. (i) Fix $\boldsymbol{\theta} \in \mathbb{R}^k$ and $\mathbf{V} \in \mathcal{M}_k$, and let $\boldsymbol{\vartheta} := (\boldsymbol{\theta}', \mathbf{0}')' \in \mathbb{R}^L$. Lemma 5.1(i) implies that $\hat{\mathbf{R}}^{(n)\mathrm{M}}_{H,\boldsymbol{\theta}} + o_\mathrm{P}(1) = \mathbf{R}^{(n)\mathrm{M}}_{H,\boldsymbol{\theta};\mathbf{V}} = \tilde{\mathbf{R}}^{(n)\mathrm{M}}_{H,\boldsymbol{\theta}} + o_\mathrm{P}(1)$ as $n \to \infty$, under $\mathrm{P}^{(n)}_{\boldsymbol{\vartheta},\mathbf{V},f}$, hence also under the contiguous sequence of alternatives $\mathrm{P}^{(n)}_{\boldsymbol{\vartheta}+n^{-1/2}\boldsymbol{\tau}^{(n)},\mathbf{V},f}$. The result is then obtained as usual, by establishing the joint normality under $\mathrm{P}^{(n)}_{\boldsymbol{\vartheta},\mathbf{V},f}$ of $\mathbf{R}^{(n)\mathrm{M}}_{H,\boldsymbol{\theta};\mathbf{V}}$ and the local log-likelihood $\log(d\mathrm{P}^{(n)}_{\boldsymbol{\vartheta}+n^{-1/2}\boldsymbol{\tau}^{(n)},\mathbf{V},f}/d\mathrm{P}^{(n)}_{\boldsymbol{\vartheta},\mathbf{V},f})$, then applying Le Cam's third Lemma (the classical Cramér-Wold device yields the required joint normality). The result for $\hat{\mathbf{R}}^{(n)\mathrm{f}}_{H,\boldsymbol{\theta}}$ follows from Lemma 5.1 in exactly the same way.

(ii) Let $\boldsymbol{\theta}$, $\mathbf{V}$, and $\boldsymbol{\vartheta}$ be as above. Lemma 5.1(i) then implies that $\hat{Q}^{(n)\mathrm{M}}_{H,\boldsymbol{\theta}} + o_\mathrm{P}(1) = k(\mathbf{R}^{(n)\mathrm{M}}_{H,\boldsymbol{\theta};\mathbf{V}})'\mathbf{R}^{(n)\mathrm{M}}_{H,\boldsymbol{\theta};\mathbf{V}} = \tilde{Q}^{(n)\mathrm{M}}_{H,\boldsymbol{\theta}} + o_\mathrm{P}(1)$ (resp., $\hat{Q}^{(n)\mathrm{f}}_{H,\boldsymbol{\theta}} = k^2(\mathbf{R}^{(n)\mathrm{f}}_{H,\boldsymbol{\theta};\mathbf{V}})'\mathbf{R}^{(n)\mathrm{f}}_{H,\boldsymbol{\theta};\mathbf{V}} + o_\mathrm{P}(1))$ as $n \to \infty$, under $\mathrm{P}^{(n)}_{\boldsymbol{\vartheta},\mathbf{V},f}$, hence also under $\mathrm{P}^{(n)}_{\boldsymbol{\vartheta}+n^{-1/2}\boldsymbol{\tau}^{(n)},\mathbf{V},f}$. As for $\tilde{Q}^{(n)\mathrm{f}}_{H,\boldsymbol{\theta}}$, Hallin and Paindaveine (2002b) actually establishes that, under strict ellipticity assumptions, we have $\tilde{Q}^{(n)\mathrm{f}}_{H,\boldsymbol{\theta}} = k^2(\mathbf{R}^{(n)\mathrm{f}}_{H,\boldsymbol{\theta};\mathbf{V}})'\mathbf{R}^{(n)\mathrm{f}}_{H,\boldsymbol{\theta};\mathbf{V}} + o_\mathrm{P}(1)$ as $n \to \infty$, under $\mathrm{P}^{(n)}_{\boldsymbol{\vartheta},\mathbf{V},f}$ (hence also under $\mathrm{P}^{(n)}_{\boldsymbol{\vartheta}+n^{-1/2}\boldsymbol{\tau}^{(n)},\mathbf{V},f}$), and it is easy to check that the proof also holds under elliptical directions. Part (i) of the proposition then yields the result for $\boldsymbol{\theta}$-specified test statistics. For the $\boldsymbol{\theta}$-unspecified ones, the result then follows from the asymptotic representation results obtained in the proof of Proposition 5.1(ii). $\square$

Proof of Proposition 5.4. Defining $\mathbf{T}^{(n)\mathcal{N},\boldsymbol{\Sigma}}_{H,\boldsymbol{\theta}} := [\mathbf{I}_H \otimes (\boldsymbol{\Sigma} \otimes \boldsymbol{\Sigma})]^{-1/2}((n-1)^{1/2}(\mathrm{vec}\,\boldsymbol{\gamma}^{(n)\mathcal{N}}_{1,\boldsymbol{\theta}})', \ldots, (n-H)^{1/2}(\mathrm{vec}\,\boldsymbol{\gamma}^{(n)\mathcal{N}}_{H,\boldsymbol{\theta}})')'$, the continuous mapping theorem shows that $Q^{(n)\mathcal{N}}_{H,\boldsymbol{\theta}} = (\mathbf{T}^{(n)\mathcal{N},\boldsymbol{\Sigma}}_{H,\boldsymbol{\theta}})'\mathbf{T}^{(n)\mathcal{N},\boldsymbol{\Sigma}}_{H,\boldsymbol{\theta}} + o_\mathrm{P}(1)$ as $n \to \infty$, under $\mathrm{P}^{(n)}_{\boldsymbol{\vartheta},\mathbf{V},f}$, hence also under the contiguous sequence of alternatives $\mathrm{P}^{(n)}_{\boldsymbol{\vartheta}+n^{-1/2}\boldsymbol{\tau}^{(n)},\mathbf{V},f}$. The result then follows, as in the previous proof, by establishing the joint normality under $\mathrm{P}^{(n)}_{\boldsymbol{\vartheta},\mathbf{V},f}$ of $\mathbf{T}^{(n)\mathcal{N},\boldsymbol{\Sigma}}_{H,\boldsymbol{\theta}}$ and $\log(d\mathrm{P}^{(n)}_{\boldsymbol{\vartheta}+n^{-1/2}\boldsymbol{\tau}^{(n)},\mathbf{V},f}/d\mathrm{P}^{(n)}_{\boldsymbol{\vartheta},\mathbf{V},f})$, then applying Le Cam's third Lemma. $\square$

Proof of Proposition 5.5. The claim follows easily by using Propositions 5.3 and 5.4 in Theorem A.1 of Nyblom and Mäkeläinen (1983). $\square$

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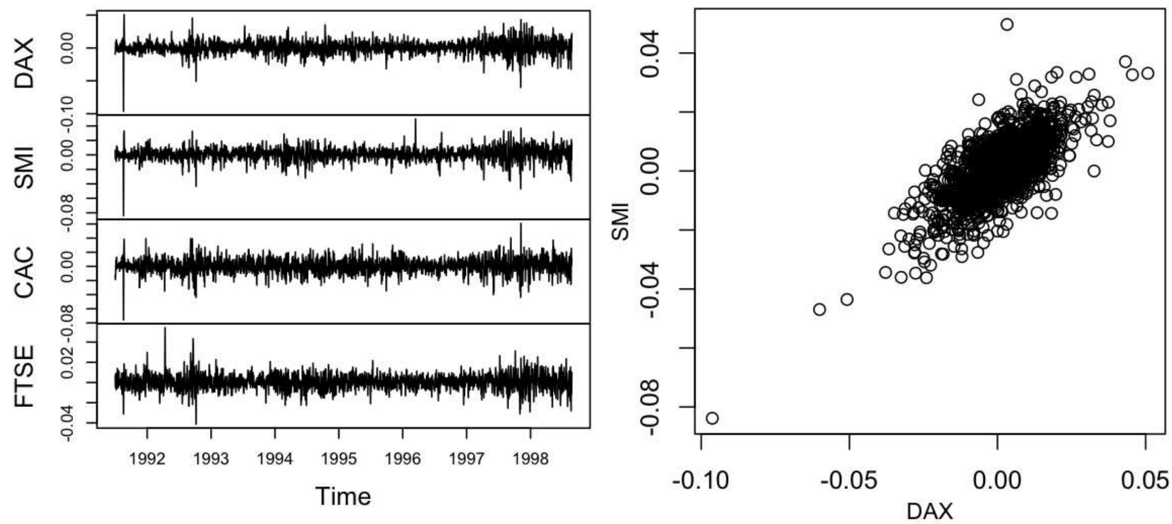

Figure 1: Log returns from 1,860 daily closing prices of four major European stock indices: Germany DAX, Switzerland SMI, France CAC, and UK FTSE (left plot). Scatter plot of SMI vs DAX log returns (right plot).

	log returns						squared log returns		
	specified $\boldsymbol{\theta}$			unspecified $\boldsymbol{\theta}$			unspecified $\boldsymbol{\theta}$		
	$\hat{\phi}^{(n)\mathrm{f}}_{H,\boldsymbol{\theta}}$	$\hat{\phi}^{(n)\mathrm{M}}_{H,\boldsymbol{\theta}}$	$\phi^{(n)\mathcal{N}}_{H,\boldsymbol{\theta}}$	$\hat{\phi}^{(n)\mathrm{f}}_{H}$	$\hat{\phi}^{(n)\mathrm{M}}_{H}$	$\phi^{(n)\mathcal{N}}_{H}$	$\hat{\phi}^{(n)\mathrm{f}}_{H}$	$\hat{\phi}^{(n)\mathrm{M}}_{H}$	$\phi^{(n)\mathcal{N}}_{H}$
$H = 1$	3e-06	4e-05	3e-08	5e-07	4e-05	4e-08	8e-04	3e-01	2e-13
$H = 2$	2e-05	4e-06	5e-07	5e-06	2e-05	6e-07	1e-05	2e-03	<1e-15
$H = 3$	1e-05	5e-06	1e-07	2e-06	1e-05	1e-07	4e-06	2e-04	<1e-15

Table 1: p-values of several shape-based runs tests and Gaussian Portmanteau tests for the series of log returns and the series of squared log returns considered in Section 4.

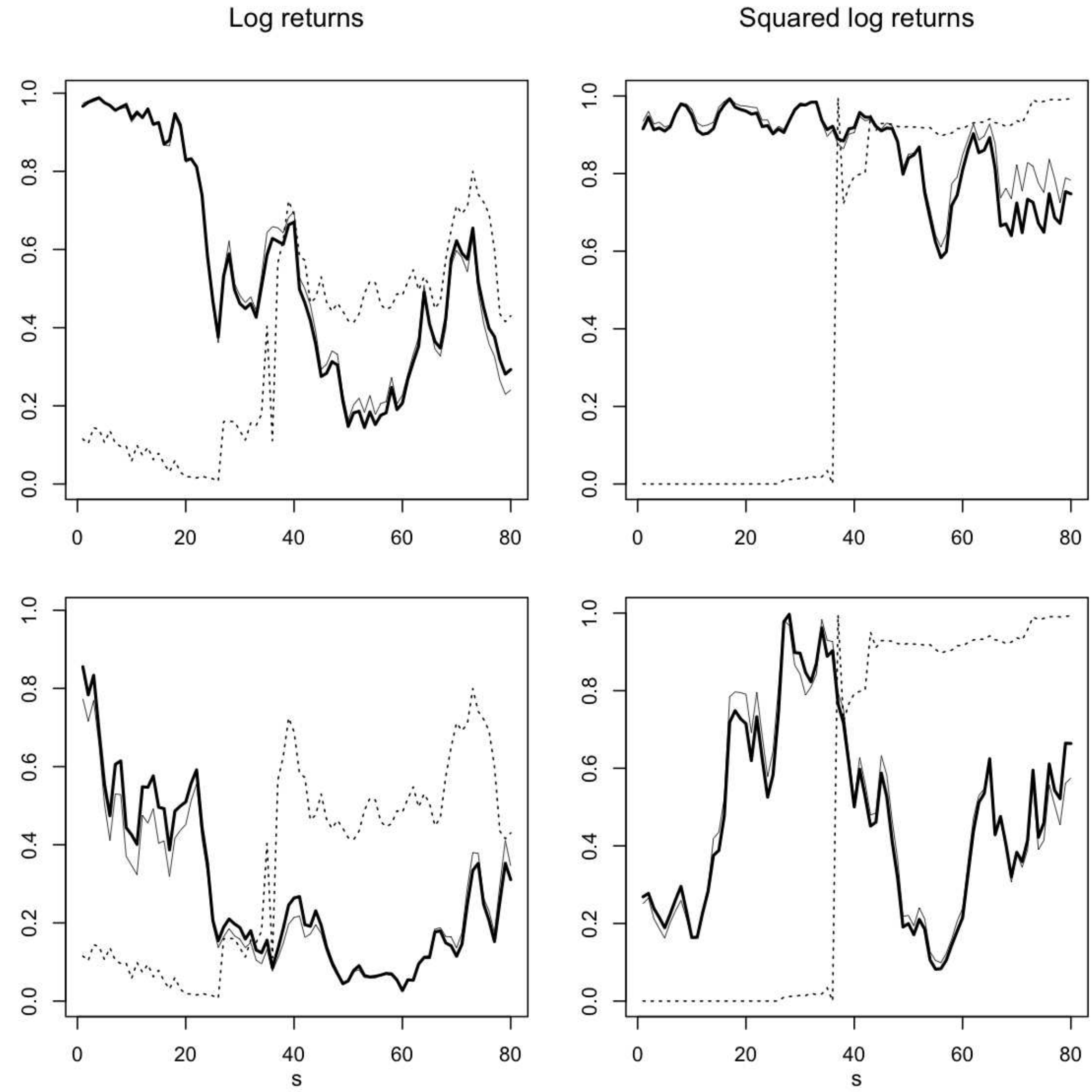

Figure 2: p-values of the shape-based/hyperplane-based full-rank runs tests $\hat{\phi}_1^{(n)\mathrm{f}}/\tilde{\phi}_1^{(n)\mathrm{f}}$ (thin/thick solid lines in the top plots), of their Marden-type counterparts $\hat{\phi}_1^{(n)\mathrm{M}}/\tilde{\phi}_1^{(n)\mathrm{M}}$ (thin/thick solid lines in the bottom plots), and of the Gaussian Portmanteau test $\phi_1^{(n)\mathcal{N}}$ (the dashed line in both top and bottom plots), in the 80 subseries associated with time indices $(s, s+1, \ldots, s+99)$, $s = 1, \ldots, 80$ (for log returns, $\boldsymbol{\theta} = \mathbf{0}$-specified tests provide plots that are extremely similar to the left plots above; see Paindaveine 2009).

		underlying radial density f									
		t_2	t_3	t_6	t_{12}	$\mathcal{N}$	e_2	e_3	e_5	e_∞	inf
	1	∞	.66	.49	.44	.41	.29	.27	.26	.25	.00
	2	∞	1.00	.75	.68	.62	.50	.47	.46	.44	.25
k	4	∞	1.27	.95	.85	.78	.69	.67	.65	.64	.56
	6	∞	1.37	1.03	.93	.85	.78	.76	.75	.73	.69
	10	∞	1.47	1.10	.99	.90	.86	.85	.84	.83	.81
	∞	—	1.62	1.22	1.10	1.00	1.00	1.00	1.00	—	1.00

Table 2: Numerical values of the functional factor $\zeta^2_{k,f}$, that is, of the AREs of the full-rank runs tests with respect to their Gaussian Portmanteau competitors, for several dimensions k and under t (with 3, 6, 12 degrees of freedom), Gaussian, and power-exponential densities (with tail parameter $\eta = 2, 3, 5$), as well as for some boundary cases. The last column provides the lower bound computed from (5.2) for each dimension k.

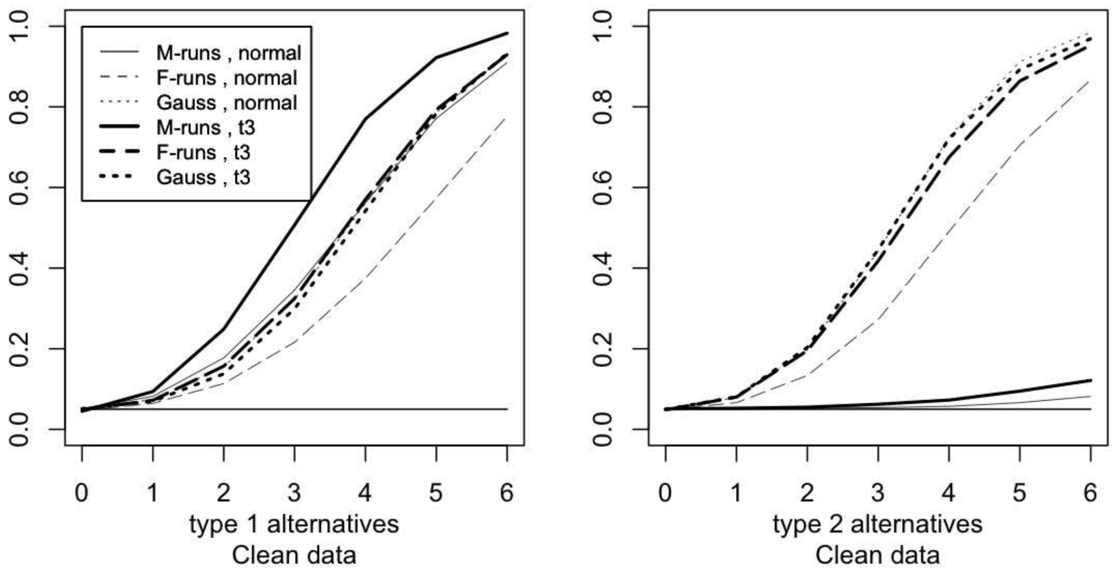

Figure 3: Rejection frequencies for Simulation (A); see Section 6 for details. *M-runs*, *F-runs*, and *Gauss* stand for *Marden-type runs tests*, *full-rank runs tests*, and *Gaussian Portmanteau tests*, respectively. Innovations are either Gaussian (*normal*) or t_3.

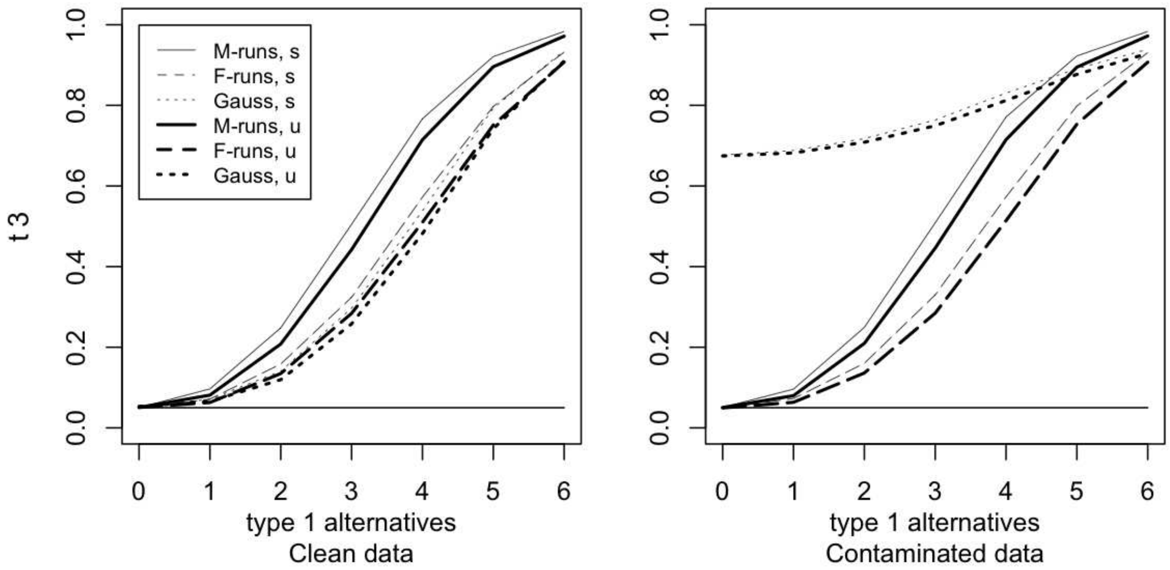

Figure 4: Rejection frequencies, for Simulation (B), of several $\boldsymbol{\theta}$-specified tests (s) and $\boldsymbol{\theta}$-unspecified tests (u); see Section 6 for details. *M-runs*, *F-runs*, and *Gauss* stand for *Marden-type runs tests*, *full-rank runs tests*, and *Gaussian Portmanteau tests*, respectively.

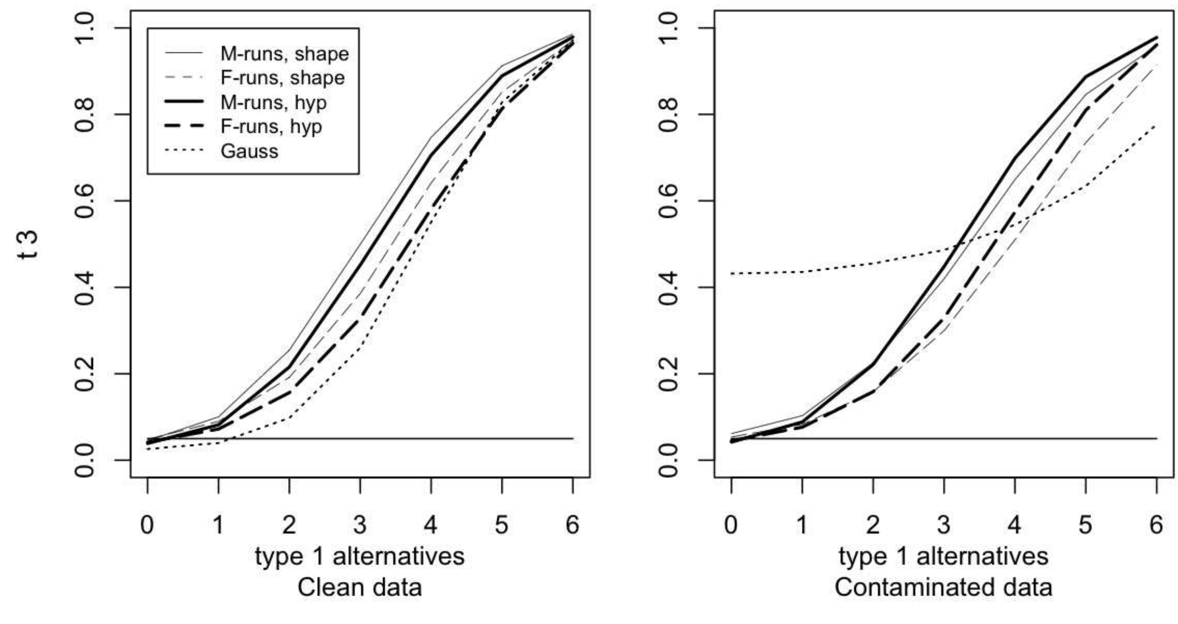

Figure 5: Rejection frequencies for Simulation (C); see Section 6 for details. *M-runs*, *F-runs*, and *Gauss* stand for *Marden-type runs tests*, *full-rank runs tests*, and *Gaussian Portmanteau tests*, respectively. Both shape-based (*shape*) and hyperplane-based (*hyp*) runs tests are considered.