Tests of concentration for low-dimensional and high-dimensional directional data

Christine Cutting, Davy Paindaveine and Thomas Verdebout

Abstract We consider asymptotic inference for the concentration of directional data. More precisely, we propose tests for concentration (i) in the low-dimensional case where the sample size n goes to infinity and the dimension p remains fixed, and (ii) in the high-dimensional case where both n and p become arbitrarily large. To the best of our knowledge, the tests we provide are the first procedures for concentration that are valid in the (n, p)-asymptotic framework. Throughout, we consider parametric FvML tests, that are guaranteed to meet asymptotically the nominal level constraint under FvML distributions only, as well as "pseudo-FvML" versions of such tests, that meet asymptotically the nominal level constraint within the whole class of rotationally symmetric distributions. We conduct a Monte-Carlo study to check our asymptotic results and to investigate the finite-sample behavior of the proposed tests.

1 Introduction

The present paper deals with directional data; that is multivariate data for which only the directions (and not the magnitudes) are measured and which therefore belong to the unit sphere $\mathscr{S}^{p-1} := \{\mathbf{x} \in \mathbb{R}^p : \|\mathbf{x}\|^2 = \mathbf{x}'\mathbf{x} = 1\}$ of $\mathbb{R}^p$. Such data arise in many different disciplines and in particular are often encountered in earth sciences such as astrophysics ([4]) and meteorology ([10]). Since the seminal paper of [11], they have been extensively studied; we refer to [17] for a general overview of the topic.

More and more applications involve data whose dimension can be large compared to the sample size. This is also the case for directional data : high-dimensional data can indeed be found in magnetic resonance

Christine Cutting
Université libre de Bruxelles, Département de Mathématique - CP 210, Boulevard du Triomphe, B-1050 Brussels, Belgium. e-mail: chcutting@ulb.ac.be

Davy Paindaveine
Université libre de Bruxelles, Département de Mathématique and ECARES, Avenue F.D. Roosevelt, 50, CP 114/04, B-1050 Brussels, Belgium. e-mail: dpaindav@ulb.ac.be. Research supported by an A.R.C. contract from the Communauté Française de Belgique and by the IAP research network grant P7/06 of the Belgian government (Belgian Science Policy).

Thomas Verdebout
Université libre de Bruxelles, Département de Mathématique - CP 210, Boulevard du Triomphe, B-1050 Brussels, Belgium. e-mail: tverdebo@ulb.ac.be

(see [9]), gene-expression (see [2]), or in text mining (see [3]). Such data cannot be analyzed via standard statistical techniques and require developing new appropriate methods. In this vein, tests of hypotheses for high-dimensional directional data have been recently proposed in [5], [6], [8], [15] and [18]. While [5], [6], [8] and [18] focussed on the null hypothesis of uniformity on high-dimensional unit spheres, [15] tackled the high-dimensional spherical location problem.

In this paper, we consider another testing problem in directional statistics, namely the problem of testing the null hypothesis that the underlying *concentration* is equal to some given value. A distributional setup where concentration has been classically considered is related to the celebrated *Fisher-von Mises-Langevin (FvML)* distributions, that have received a lot of attention in the literature; see, e.g., Sections 10.4-10.6 in [17]. FvML distributions on $\mathscr{S}^{p-1}$ admit probability density functions (with respect to the surface area measure) that are of the form

$$\mathbf{x} \to f(\mathbf{x}) := c_{p,\kappa} \exp(\kappa \mathbf{x}'\boldsymbol{\theta}),$$

where $c_{p,\kappa} (> 0)$ is a normalization constant, $\boldsymbol{\theta} \in \mathscr{S}^{p-1}$ is a location parameter, and $\kappa (> 0)$ is a concentration parameter. The larger the value of κ, the more concentrated about $\boldsymbol{\theta}$ the distribution is. In the fixed-p case, the problem of developing inferential procedures on $\boldsymbol{\theta}$ and/or κ has been extensively studied in the literature. When testing $\mathscr{H}_0 : \boldsymbol{\theta} = \boldsymbol{\theta}_0$ against $\mathscr{H}_1 : \boldsymbol{\theta} \neq \boldsymbol{\theta}_0$, for instance, one of the most classical tests is the score test from Watson [24]. This test was shown in [19] to be locally and asymptotically optimal, and is furthermore robust to high-dimensionality (see [15]).

Besides the tests described in [17], tests of hypotheses that specifically address problems on the concentration parameter can mainly be found in [14], [21] and [23]. These tests are fixed-p FvML likelihood ratio or score tests. Such tests are asymptotically efficient in the FvML case, but are not robust to departures from FvML distributions (as we explain in Section 2, concentration can be defined away from the FvML case). Fixed-p robust procedures for concentration have therefore been proposed by [12] and [13] in the one-sample case and recently by [22] in the multi-sample case. In all cases, however, fixed-p tests for concentration fail to be robust to high-dimensionality. The objective of the present paper is therefore to provide high-dimensional tests for concentration.

The paper is organized as follows. In Section 2, we first define the problem of testing for concentration. Then we propose a new robust fixed-p test and investigate its asymptotic properties. In Section 3, we develop a high-dimensional test for concentration and we study its (n, p)-asymptotic properties under the null hypothesis. Finally, in Section 4, we conduct low-dimensional and high-dimensional Monte-Carlo simulations to confirm our theoretical results and investigate the finite-sample properties of the proposed tests.

2 Testing for concentration in low dimensions

Let $\mathbf{X}_1, \ldots, \mathbf{X}_n$ be independent random p-vectors sharing an FvML distribution with location $\boldsymbol{\theta}$ and concentration κ. We consider the problem of testing the null hypothesis $\mathscr{H}_0 : \kappa = \kappa_0$ against $\mathscr{H}_1 : \kappa \neq \kappa_0$, where $\kappa_0 > 0$ is fixed. Of course, κ is then the parameter of interest, while $\boldsymbol{\theta}$ plays the role of a nuisance parameter. The null hypothesis $\mathscr{H}_0$ is clearly invariant with respect to the group of rotations, so that the invariance principle yields to resorting to tests that are invariant under this group. Since the group of rotations is actually generating the null hypothesis $\mathscr{H}_0$, invariant tests are distribution-free under $\mathscr{H}_0$. All tests we will consider in this paper are invariant, so that we may throughout, without any loss of generality, restrict to the case where $\boldsymbol{\theta}$ coincides with the first vector of the canonical basis of $\mathbb{R}^p$.

Denoting by $I_\nu(\cdot)$ the order-ν modified Bessel function of the first kind, it is easy to show that

$$e_1 := \mathrm{E}[\mathbf{X}_i'\boldsymbol{\theta}] = h_p(\kappa), \qquad i = 1, \ldots, n, \tag{1}$$

where the mapping

$$\begin{aligned} h_p : \mathbb{R}^+ &\to (0,1) \\ z &\mapsto \frac{I_{p/2}(z)}{I_{p/2-1}(z)} \end{aligned} \tag{2}$$

is one-to-one. Consequently, concentration, for fixed-p, may equivalently be measured through e_1, and one may rephrase the null hypothesis $\mathscr{H}_0 : \kappa = \kappa_0$ as $\mathscr{H}_0 : e_1 = e_{10}$, with $e_{10} := h_p(\kappa_0)$. In the sequel, we rather adopt the latter formulation of the null hypothesis, since this formulation, unlike the former, makes sense away from the FvML case.

As mentioned in the introduction, the tests for concentration available in the literature are mainly of a likelihood ratio or score nature. The most classical test for the null hypothesis $\mathscr{H}_0 : e_1 = e_{10}$ is the Watamori and Jupp ([23]) score test $\phi_{\mathrm{WJ}}^{(n)}$ that rejects the null hypothesis at asymptotic level α whenever

$$T_{\mathrm{WJ}}^{(n)} := \frac{n(\|\bar{\mathbf{X}}_n\| - e_{10})^2}{1 - \frac{p-1}{\kappa_0}e_{10} - e_{10}^2} > \chi^2_{1,1-\alpha},$$

where $\bar{\mathbf{X}}_n := n^{-1}\sum_{i=1}^n \mathbf{X}_i$ and $\chi^2_{\ell,1-\alpha}$ stands for the α-upper quantile of the chi-square distribution with ℓ degrees of freedom. This test is asymptotically equivalent to the corresponding FvML likelihood ratio test, hence is locally and asymptotically optimal in the FvML case; see [16]. Because of its parametric nature, however, $\phi_{\mathrm{WJ}}^{(n)}$ relies crucially on the FvML assumption, in the sense that there is no guarantee that it meets the asymptotic level constraint away from the FvML case.

In this section, we show that an appropriate robustification of $\phi_{\mathrm{WJ}}^{(n)}$ is valid under the class of *rotationally symmetric* distributions. A random vector $\mathbf{X}$, taking values on the unit sphere $\mathscr{S}^{p-1}$ of $\mathbb{R}^p$, is said to be *rotationally symmetric* about $\boldsymbol{\theta}(\in \mathscr{S}^{p-1})$ if and only if, for all orthogonal $p \times p$ matrices $\mathbf{O}$ satisfying $\mathbf{O}\boldsymbol{\theta} = \boldsymbol{\theta}$, the random vectors $\mathbf{OX}$ and $\mathbf{X}$ are equal in distribution. If, further, $\mathbf{X}$ is absolutely continuous (still with respect to the surface area measure on $\mathscr{S}^{p-1}$), then the corresponding density is of the form

$$\mathbf{x} \to c_{p,f}\, f(\mathbf{x}'\boldsymbol{\theta}), \tag{3}$$

where $c_{p,f}(> 0)$ is a normalization constant and $f : [-1,1] \to \mathbb{R}$ is some nonnegative function. In the general (possibly non-absolutely continuous) case, rotationally symmetric distributions are characterized by the location parameter $\boldsymbol{\theta}$ and the cumulative distribution function F of $\mathbf{X}'\boldsymbol{\theta}$; such distributions are therefore of a semiparametric nature. The rotationally symmetric distribution associated with $\boldsymbol{\theta}$ and F will be denoted as $\mathscr{R}_p(\boldsymbol{\theta}, F)$. For identifiability purposes, it will be tacitly assumed throughout that F belongs to the collection $\mathscr{F}$ of cumulative distribution functions $F : [-1,1] \to [0,1]$ such that $e_1 = \mathrm{E}[\mathbf{X}'\boldsymbol{\theta}] > 0$ (the assumption that $e_1 \neq 0$ makes the pair $\{\pm\boldsymbol{\theta}\}$ identifiable and imposing further that $e_1 > 0$ makes $\boldsymbol{\theta}$ itself identifiable). When a null hypothesis of the form $\mathscr{H}: e_1 = e_{10}$ is considered, $\mathscr{F}_0$ will stand for the subset of $\mathscr{F}$ corresponding to the null hypothesis.

FvML distributions are (absolutely continuous) rotationally symmetric distributions, and correspond to $f(t) = \exp(\kappa t)$, or, equivalently, to

$$F_{p,\kappa}(t) = c_{p,\kappa} \int_{-1}^{t} (1-s^2)^{(p-3)/2} \exp(\kappa s)\, ds \qquad (t \in [-1,1]),$$

where $c_{p,\kappa}$ is the same normalization constant as in the introduction. According to the equivalence between κ and e_1 in (1)-(2), the FvML cumulative distribution function $F_{p,\kappa}$ belongs to $\mathscr{F}$ (resp., to $\mathscr{F}_0$) if and only if $\kappa > 0$ (resp., if and only if $\kappa = \kappa_0 := h_p^{-1}(e_{10})$).

Assume now that a random sample $\mathbf{X}_1, \ldots, \mathbf{X}_n$ from a rotationally symmetric distribution is available. We then consider the robustified test $\phi^{(n)}_{\rm WJm}$ that rejects the null hypothesis $\mathscr{H}_0: e_1 = e_{10}$ at asymptotic level α whenever

$$T^{(n)}_{\rm WJm} := \frac{n(\|\bar{\mathbf{X}}_n\| - e_{10})^2}{\hat{e}_{n2} - e_{10}^2} > \chi^2_{1,1-\alpha},$$

where we let $\hat{e}_{n2} := \bar{\mathbf{X}}_n' \mathbf{S}_n \bar{\mathbf{X}}_n / \|\bar{\mathbf{X}}_n\|^2$, with $\mathbf{S}_n := n^{-1} \sum_{i=1}^n \mathbf{X}_i \mathbf{X}_i'$. In the FvML case, $\phi^{(n)}_{\rm WJm}$ is asymptotically equivalent to $\phi^{(n)}_{\rm WJ}$ under the null hypothesis (hence also under sequences of contiguous alternatives), but $\phi^{(n)}_{\rm WJm}$ is further asymptotically valid (in the sense that it meets asymptotically the nominal level constraint) under any rotationally symmetric distribution. This is made precise in the following result (see the appendix for a proof).

Theorem 1. *Fix $p \in \{2,3,\ldots\}$, $\boldsymbol{\theta} \in \mathscr{S}^{p-1}$, and $F \in \mathscr{F}_0$, and denote by $\mathscr{R}_p^{(n)}(\boldsymbol{\theta}, F)$ the hypothesis under which the random p-vectors $\mathbf{X}_1, \ldots, \mathbf{X}_n$ are mutually independent and share the distribution $\mathscr{R}_p(\boldsymbol{\theta}, F)$. Then, (i) under $\mathscr{R}_p^{(n)}(\boldsymbol{\theta}, F)$, $T^{(n)}_{\rm WJm}$ converges weakly to the χ^2_1 distribution as $n \to \infty$; (ii) under $\mathscr{R}_p^{(n)}(\boldsymbol{\theta}, F_{p,\kappa_0})$, with $\kappa_0 = h_p^{-1}(e_{10})$, $T^{(n)}_{\rm WJm} - T^{(n)}_{\rm WJ} = o_{\rm P}(1)$ as $n \to \infty$, so that $\phi^{(n)}_{\rm WJm}$ is locally and asymptotically optimal in the FvML case.*

This result shows that the robustified test $\phi^{(n)}_{\rm WJm}$ enjoys nice properties. Like any fixed-p test, however, it requires the sample size n to be large compared to the dimension p. Figure 1 below indeed confirms that, parallel to the classical test $\phi^{(n)}_{\rm WJ}$, the robustified test $\phi^{(n)}_{\rm WJm}$ fails to maintain the proper null size in high dimensions. In the next section, we therefore define high-dimensional tests for concentration.

3 Testing for concentration in high dimensions

3.1 The FvML case

We start with the high-dimensional FvML case. To this end, it is natural to consider triangular arrays of observations $\mathbf{X}_{ni}$, $i = 1, \ldots, n$, $n = 1, 2, \ldots$ such that, for any n, the FvML random vectors $\mathbf{X}_{n1}, \mathbf{X}_{n2}, \ldots, \mathbf{X}_{nn}$ are mutually independent from $\mathscr{R}_{p_n}(\boldsymbol{\theta}_n, F_{p_n,\kappa})$, where the sequence (p_n) goes to infinity with n and where $\boldsymbol{\theta}_n \in \mathscr{S}^{p_n - 1}$ for any n (we will denote the resulting hypothesis as $\mathscr{R}_{p_n}(\boldsymbol{\theta}_n, F_{p_n,\kappa})$). In the present high-dimensional framework, however, considering a fixed, that is p-independent, value of κ is not appropriate. Indeed, for any fixed $\kappa > 0$, Proposition 1(i) below shows that $\mathbf{X}_{ni}' \boldsymbol{\theta}_n$, under $\mathscr{R}_{p_n}^{(n)}(\boldsymbol{\theta}_n, F_{p_n,\kappa})$,

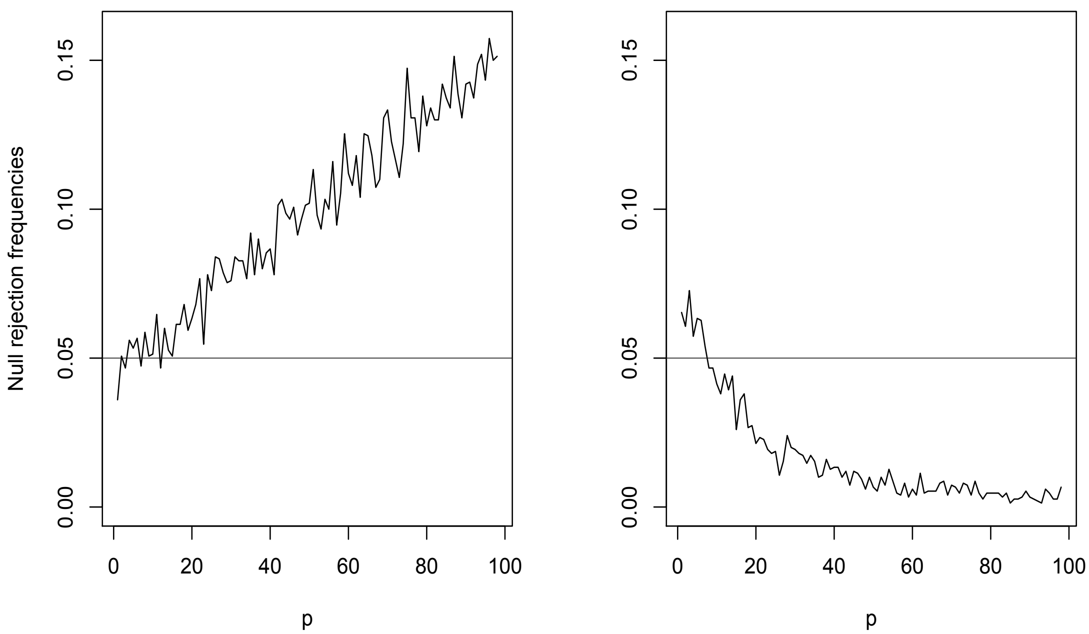

Fig. 1 For any $p = 2, 3, \ldots, 100$, the left panel reports null rejection frequencies of the fixed-p FvML test $\phi^{(n)}_{\mathrm{WJ}}$ for $\mathscr{H}_0 : \kappa = p$ (at nominal level 5%), obtained from $M = 1{,}500$ independent random samples of size $n = 100$ from the FvML distribution with a location $\boldsymbol{\theta}$ equal to the first vector of the canonical basis of $\mathbb{R}^p$. The right panel reports the corresponding rejection frequencies of the robustified test $\phi^{(n)}_{\mathrm{WJm}}$.

converges in quadratic mean to zero. In other words, irrespective of the value of κ, the sequence of FvML distributions considered eventually puts mass on the "equator" $\{\mathbf{x} \in \mathscr{S}^{p_n-1} : \mathbf{x}'\boldsymbol{\theta}_n = 0\}$ only, which leads to a common concentration scheme across κ-values. For p-independent κ-values, the problem of testing $\mathscr{H}_0 : \kappa = \kappa_0$ versus $\mathscr{H}_1 : \kappa \neq \kappa_0$ for a given κ_0 is therefore ill-posed in high dimensions.

We then rather consider null hypotheses of the form $\mathscr{H}_0 : e_{n1} = e_{10}$, where we let $e_{n1} := \mathrm{E}[\mathbf{X}'_{n1}\boldsymbol{\theta}_n]$ and where $e_{10} \in (0,1)$ is fixed. Such hypotheses, in the FvML case, are associated with triangular arrays as above but where the concentration parameter κ assumes a value that depends on n in an appropriate way. The following result makes precise the delicate relation between the resulting concentration sequence κ_n and the alternative concentration parameter e_{1n} in the high-dimensional case (see the appendix for a proof).

Proposition 1. *Let (p_n) be a sequence of positive integers diverging to ∞, $(\boldsymbol{\theta}_n)$ be an arbitrary sequence such that $\boldsymbol{\theta}_n \in \mathscr{S}^{p_n-1}$ for any n, and (κ_n) be a sequence in $(0,\infty)$. Under the resulting sequence of hypotheses $\mathscr{R}^{(n)}_{p_n}(\boldsymbol{\theta}_n, F_{p_n,\kappa_n})$, write $e_{n1} := \mathrm{E}[\mathbf{X}'_{n1}\boldsymbol{\theta}_n]$ and $\tilde{e}_{n2} := \mathrm{Var}[\mathbf{X}'_{n1}\boldsymbol{\theta}_n]$. Then we have the following (where all convergences are as $n \to \infty$) :*
(i) $\kappa_n/p_n \to 0 \Leftrightarrow e_{n1} \to 0$;
(ii) $\kappa_n/p_n \to c \in (0,\infty) \Leftrightarrow e_{n1} \to g_1(c)$, *where* $g_1 : (0,\infty) \to (0,1) : x \mapsto x/(\frac{1}{2} + (x^2 + \frac{1}{4})^{1/2})$;
(iii) $\kappa_n/p_n \to \infty \Leftrightarrow e_{n1} \to 1$.
In cases (i) and (iii), $\tilde{e}_{n2} \to 0$, whereas in case (ii), $\tilde{e}_{n2} \to g_2(c)$, for some function $g_2 : (0,\infty) \to (0,1)$.

Parts (i) and (iii) of this proposition are associated with the null hypotheses $\mathscr{H}_0 : e_{n1} = 0$ and $\mathscr{H}_0 : e_{n1} = 1$, respectively. The former null hypothesis has already been addressed in [8], while the latter is extremely pathological since it corresponds to distributions that put mass on a single point on the sphere, namely $\boldsymbol{\theta}_n$. As already announced above, we therefore focus throughout on the null hypothesis $\mathscr{H}_0 : e_{n1} = e_{10}$, where $e_{10} \in (0,1)$ is fixed. Part (ii) of Proposition 1 shows that, in the FvML case, this can be obtained only when κ_n goes to infinity at the same rate as p_n; more precisely, the null hypothesis $\mathscr{H}_0 : e_{n1} = e_{10}$ is associated with sequences (κ_n) such that $\kappa_n/p_n \to c_0$, with $c_0 = g_1^{-1}(e_{10})$.

As shown in Figure 1, the fixed-p tests $\phi^{(n)}_{\rm WJ}/\phi^{(n)}_{\rm WJm}$ fail to be robust to high-dimensionality, which calls for corresponding high-dimensional tests. The following result, that is proved in the appendix, shows that, in the FvML case, such a high-dimensional test is the test $\phi^{(n)}_{\rm CPV}$ that rejects $\mathscr{H}_0 : e_{n1} = e_{10}$ whenever

$$|Q^{(n)}_{\rm CPV}| > z_{\alpha/2},$$

where

$$Q^{(n)}_{\rm CPV} := \frac{\sqrt{p_n}\big(n\|\bar{\mathbf{X}}_n\|^2 - 1 - (n-1)e_{10}^2\big)}{\sqrt{2}\big(p_n\big(1-\frac{e_{10}}{c_0}-e_{10}^2\big)^2 + 2np_ne_{10}^2\big(1-\frac{e_{10}}{c_0}-e_{10}^2\big)+\big(\frac{e_{10}}{c_0}\big)^2\big)^{1/2}}, \quad \text{with } c_0 = g_1^{-1}(e_{10}),$$

and where z_β stands for the β-upper quantile of the standard normal distribution.

Theorem 2. *Let (p_n) be a sequence of positive integers diverging to ∞, $(\boldsymbol{\theta}_n)$ be an arbitrary sequence such that $\boldsymbol{\theta}_n \in \mathscr{S}^{p_n-1}$ for any n, and (κ_n) be a sequence in $(0,\infty)$ such that, for any n, $e_{n1} = e_{10}$ under $\mathscr{R}^{(n)}_{p_n}(\boldsymbol{\theta}_n, F_{p_n,\kappa_n})$. Then, under the sequence of hypotheses $\mathscr{R}^{(n)}_{p_n}(\boldsymbol{\theta}_n, F_{p_n,\kappa_n})$, $Q^{(n)}_{\rm CPV}$ converges weakly to the standard normal distribution as $n \to \infty$.*

As in the fixed-p case, the test $\phi^{(n)}_{\rm CPV}$ is a parametric test whose (n,p)-asymptotic validity requires stringent FvML assumptions. In the next section, we therefore propose a robustified version of this test, that is robust to both high-dimensionality and departures from the FvML case.

3.2 The general rotationally symmetric case

We intend to define a high-dimensional test for concentration that is valid in the general rotationally symmetric case. To this end, consider triangular arrays of observations $\mathbf{X}_{ni}$, $i = 1,\ldots,n$, $n = 1,2,\ldots$ such that, for any n, the random p_n-vectors $\mathbf{X}_{n1}, \mathbf{X}_{n2}, \ldots, \mathbf{X}_{nn}$ are mutually independent and share a rotationally symmetric distribution with location parameter $\boldsymbol{\theta}_n$ and cumulative distribution F_n, where the sequence (p_n) goes to infinity with n and where $\boldsymbol{\theta}_n \in \mathscr{S}^{p_n-1}$ for any n (in line with Section 2, F_n is the cumulative distribution function of $\mathbf{X}'_{n1}\boldsymbol{\theta}_n$). As above, the corresponding hypothesis will be denoted as $\mathscr{R}^{(n)}_{p_n}(\boldsymbol{\theta}_n, F_n)$.

As in the FvML case, we consider the problem of testing the null hypothesis $\mathscr{H}_0 : e_{n1} = e_{10}$, where $e_{10} \in (0,1)$ is fixed. In the present rotationally symmetric case, we propose a robustified version of the test $\phi^{(n)}_{\rm CPV}$ above. This robustified test, $\phi^{(n)}_{\rm CPVm}$ say, rejects the null hypothesis at asymptotic level α whenever

$$|Q^{(n)}_{\rm CPVm}| > z_{\alpha/2},$$

where

$$Q_{\mathrm{CPVm}}^{(n)} := \frac{\sqrt{p_n}\big(n\|\bar{\mathbf{X}}_n\|^2 - 1 - (n-1)e_{10}^2\big)}{\sqrt{2}\big(p_n\big(\hat{e}_{n2} - \|\bar{\mathbf{X}}_n\|^2\big)^2 + 2np_n e_{10}^2\big(\hat{e}_{n2} - \|\bar{\mathbf{X}}_n\|^2\big) + (1-\hat{e}_{n2})^2\big)^{1/2}};$$

recall from Section 2 that $\hat{e}_{n2} = \bar{\mathbf{X}}_n'\mathbf{S}_n\bar{\mathbf{X}}_n/\|\bar{\mathbf{X}}_n\|^2$, with $\mathbf{S}_n := n^{-1}\sum_{i=1}^n \mathbf{X}_i\mathbf{X}_i'$. The following result shows that, under mild assumptions, this test is asymptotically valid in the general rotationally symmetric case (see the appendix for a proof).

Theorem 3. *Let (p_n) be a sequence of positive integers diverging to ∞, and $(\boldsymbol{\theta}_n)$ be an arbitrary sequence such that $\boldsymbol{\theta}_n \in \mathscr{S}^{p_n-1}$ for any n. Let (F_n) be a sequence of cumulative distribution functions over $[-1,1]$ such that, under $\mathscr{R}_{p_n}^{(n)}(\boldsymbol{\theta}_n, F_n)$, one has $e_{n1} = e_{10}$ for any n, and*

$$(i)\ n\tilde{e}_{n2} \to \infty, \quad (ii)\ \min\Big(\frac{p_n\tilde{e}_{n2}^2}{f_{n2}^2}, \frac{\tilde{e}_{n2}}{n}\Big) = o(1), \quad (iii)\ \tilde{e}_{n4}/\tilde{e}_{n2}^2 = o(n), \quad \textit{and} \quad (iv)\ f_{n4}/f_{n2}^2 = o(n), \qquad (4)$$

where we let $\tilde{e}_{n\ell} := \mathrm{E}[(\mathbf{X}_{ni}'\boldsymbol{\theta}_n - e_{n1})^\ell]$ and $f_{n\ell} := \mathrm{E}[(1-(\mathbf{X}_{ni}'\boldsymbol{\theta}_n)^2)^{\ell/2}]$. Then, under the sequence of hypotheses $\mathscr{R}_{p_n}^{(n)}(\boldsymbol{\theta}_n, F_n)$, $Q_{\mathrm{CPVm}}^{(n)}$ converges weakly to the standard normal distribution as $n \to \infty$.

As explained in [8], Conditions (ii)-(iv) are extremely mild. In particular, they hold in the FvML case, irrespective of the sequences (κ_n) and (p_n) considered, provided, of course, that $p_n \to \infty$ as $n \to \infty$. Condition (i) is a little more restrictive. In the FvML case, for instance, it imposes that $p_n/n = o(1)$ as $n \to \infty$. Such a restriction originates in the need to estimate the quantity $\tilde{e}_{n2}$, which itself requires estimating $\boldsymbol{\theta}_n$ in an appropriate way.

4 Simulations

In this section, our objective is to study the small-sample behavior of the tests proposed in this paper. More precisely, we investigate whether or not the asymptotic critical values, for moderate-to-large sample sizes n (and dimensions p, in the high-dimensional case), lead to null rejection frequencies that are close to the nominal level.

4.1 The low-dimensional case

We first consider the low-dimensional case. For each combination of $\kappa \in \{1,3\}$ and $p \in \{3,4,5\}$, we generated $M = 2{,}500$ independent random samples $\mathbf{X}_1, \ldots, \mathbf{X}_n$ of size $n = 50$ from the Purkayastha rotationally symmetric distribution $\mathscr{R}_p(\boldsymbol{\theta}, G_{p,\kappa})$, based on

$$G_{p,\kappa}(t) = d_{p,\kappa}\int_{-1}^{t}(1-s^2)^{(p-3)/2}\exp(-\kappa\arccos(s))\,ds \qquad (t \in [-1,1]),$$

where $d_{p,\kappa}$ is a normalizing constant; for $\boldsymbol{\theta}$, we took the first vector of the canonical basis of $\mathbb{R}^p$. In each case, we considered the testing problem $\mathscr{H}_0 : e_1 = e_{10}$ vs $\mathscr{H}_1 : e_1 \neq e_{10}$, where e_{10} is taken as the underlying value of $\mathrm{E}[\mathbf{X}_1'\boldsymbol{\theta}]$ (which depends on n and p). On each sample generated above, we then performed (i) the FvML test $\phi_{\mathrm{WJ}}^{(n)}$ and (ii) its robustified version $\phi_{\mathrm{WJm}}^{(n)}$, both at nominal level 5%. Figure 2 provides the resulting empirical — by construction, null — rejection frequencies. Inspection of this figure reveals that, unlike the FvML test $\phi_{\mathrm{WJ}}^{(n)}$, the robustified test $\phi_{\mathrm{WJm}}^{(n)}$ meets the level constraint in all cases.

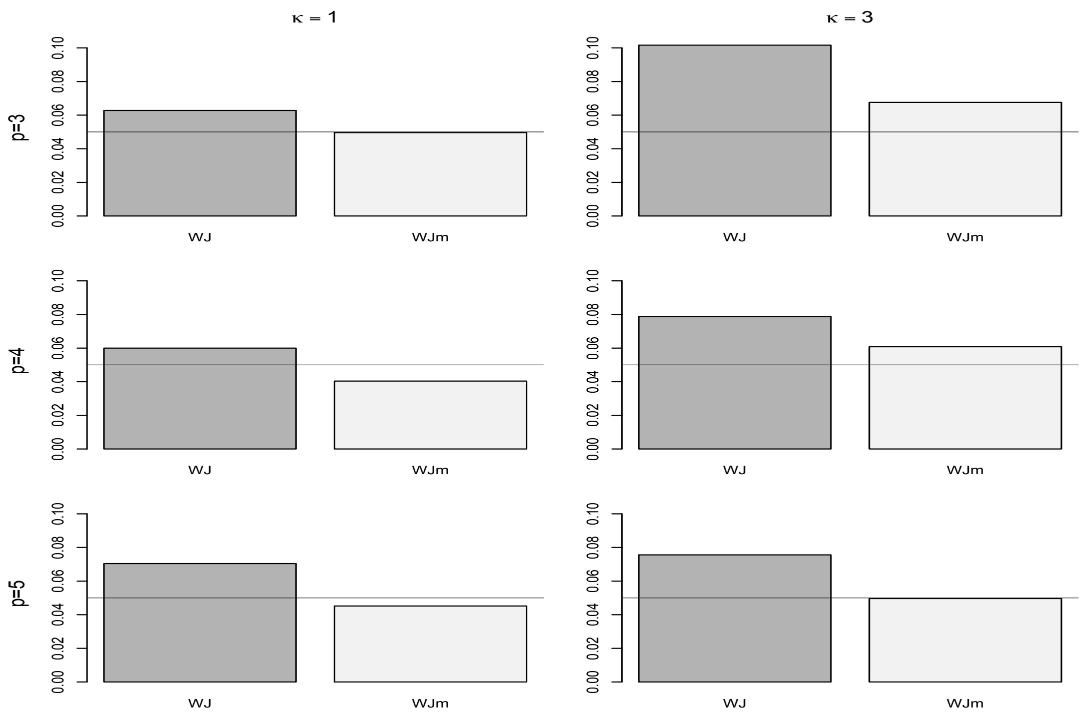

Fig. 2 Empirical null rejection frequencies of (i) the low-dimensional FvML test $\phi_{\mathrm{WJ}}^{(n)}$ and of (ii) its robustified version $\phi_{\mathrm{WJm}}^{(n)}$, under various p-dimensional Purkayastha rotationally symmetric distributions involving two different concentrations κ. Rejection frequencies are obtained from $2,500$ independent samples of size 50, and all tests are performed at asymptotic level 5%; see Section 4.1 for details.

4.2 The high-dimensional case

To investigate the behavior of the proposed high-dimensional tests, we performed two simulations. In the first one, we generated, for every $(n,p) \in C_1 \times C_1$, with $C_1 = \{30,100,400\}$, $M = 2{,}500$ independent random samples of size n from the FvML distributions $\mathscr{R}_p(\boldsymbol{\theta}, F_{p,\kappa})$, where $\boldsymbol{\theta}$ is the first vector of the canonical basis of $\mathbb{R}^p$ and where we took $\kappa = p$. In the second simulation, we generated, for every $(n,p) \in C_2 \times C_2$, with $C_2 = \{30,100\}$, $M = 2{,}500$ independent random samples of size n from the Purkayastha distributions $\mathscr{R}_p(\boldsymbol{\theta}, G_{p,\kappa})$, still with $\kappa = p$ and the same $\boldsymbol{\theta}$ as above. The Purkayastha distribution is numerically hard to generate for dimensions larger than 150, which is the only reason why the dimensions considered in this second simulation are smaller than in the first one.

Parallel to the simulations conducted for fixed p, we considered the testing problem $\mathscr{H}_0 : e_1 = e_{10}$ vs $\mathscr{H}_1 : e_1 \neq e_{10}$, where e_{10} is the underlying value of $\mathrm{E}[\mathbf{X}_1'\boldsymbol{\theta}]$. On all samples that were generated, we then performed the four following tests at nominal level 5%: (i) the low-dimensional FvML test $\phi^{(n)}_{\rm WJ}$, (ii) its robustified version $\phi^{(n)}_{\rm WJm}$, (iii) the high-dimensional FvML test $\phi^{(n)}_{\rm CPV}$, and (iv) its robustified version $\phi^{(n)}_{\rm CPVm}$. The resulting empirical (null) rejection frequencies are provided in Figures 3-4, for the FvML and Purkayastha cases, respectively. The results show that

(a) the low-dimensional tests $\phi^{(n)}_{\rm WJ}$ and $\phi^{(n)}_{\rm WJm}$ clearly fail to be robust to high-dimensionality;
(b) at the FvML, $\phi^{(n)}_{\rm CPV}$ is asymptotically valid when n and p are moderate to large;
(c) away from the FvML, the high-dimensional test $\phi^{(n)}_{\rm CPV}$ is not valid, but its robustified version $Q^{(n)}_{\rm CPVm}$ is when $n \geq p$.

In order to illustrate the asymptotic normality result in Theorems 2-3, we computed, for each (n,p) configuration and each distribution considered (FvML or Purkayastha), kernel estimators for the densities of $Q^{(n)}_{\rm CPV}$ and $Q^{(n)}_{\rm CPVm}$, based on the various collections of $2,500$ values of these test statistics obtained above. In all cases, we used Gaussian kernels with a bandwidth obtained from the "rule of thumb" in [20]. The resulting kernel density estimators are plotted in Figures 5-6, for FvML and Purkayastha distributions, respectively. Clearly, Figure 5 supports the results that both test statistics are asymptotically standard normal under the null hypothesis, whereas Figure 6 illustrates that this asymptotic behavior still holds for $Q^{(n)}_{\rm CPVm}$ (but not for $Q^{(n)}_{\rm CPV}$) away from the FvML case.

Appendix

Proof of Theorem 1. (i) All expectations and variances when proving Part (i) of the theorem are taken under $\mathscr{R}^{(n)}_p(\boldsymbol{\theta},F)$ and all stochastic convergences are taken as $n\to\infty$ under $\mathscr{R}^{(n)}_p(\boldsymbol{\theta},F)$. Since

$$n^{1/2}(\bar{\mathbf{X}}_n - e_{10}\boldsymbol{\theta}) = O_{\rm P}(1), \tag{5}$$

the delta method (applied to the mapping $\mathbf{x}\mapsto \mathbf{x}/\|\mathbf{x}\|$) yields

$$n^{1/2}(\mathbf{Y}_n - \boldsymbol{\theta}) = e_{10}^{-1}[\mathbf{I}_p - \boldsymbol{\theta}\boldsymbol{\theta}']n^{1/2}(\bar{\mathbf{X}}_n - e_{10}\boldsymbol{\theta}) + o_{\rm P}(1), \tag{6}$$

where we wrote $\mathbf{Y}_n := \bar{\mathbf{X}}_n/\|\bar{\mathbf{X}}_n\|$. This, and the fact that

$$\mathbf{S}_n \xrightarrow{\rm P} {\rm E}[\mathbf{X}_1\mathbf{X}_1'] = {\rm E}[(\mathbf{X}_1'\boldsymbol{\theta})^2]\boldsymbol{\theta}\boldsymbol{\theta}' + \frac{1-{\rm E}[(\mathbf{X}_1'\boldsymbol{\theta})^2]}{p-1}(\mathbf{I}_p - \boldsymbol{\theta}\boldsymbol{\theta}'),$$

where $\mathbf{I}_p$ denotes the p-dimensional identity matrix, readily implies that

$$\hat{\sigma}_n^2 := \frac{\bar{\mathbf{X}}_n'\mathbf{S}_n\bar{\mathbf{X}}_n}{\|\bar{\mathbf{X}}_n\|^2} - e_{10}^2 = \mathbf{Y}_n'\mathbf{S}_n\mathbf{Y}_n - e_{10}^2 \xrightarrow{\rm P} {\rm E}[(\mathbf{X}_1'\boldsymbol{\theta})^2] - e_{10}^2 = {\rm Var}[\mathbf{X}_1'\boldsymbol{\theta}]. \tag{7}$$

Now, write

$$\frac{n^{1/2}(\|\bar{\mathbf{X}}_n\| - e_{10})}{\hat{\sigma}_n} = \frac{n^{1/2}\bar{\mathbf{X}}_n'(\mathbf{Y}_n - \boldsymbol{\theta})}{\hat{\sigma}_n} + \frac{n^{1/2}(\bar{\mathbf{X}}_n'\boldsymbol{\theta} - e_{10})}{\hat{\sigma}_n} =: S_{1n} + S_{2n}, \tag{8}$$

say. It directly follows from (5)-(7) that $S_{1n} = o_{\rm P}(1)$ as $n\to\infty$. As for S_{2n}, the central limit theorem and Slutsky's lemma yield that S_{2n} is asymptotically standard normal. This readily implies that

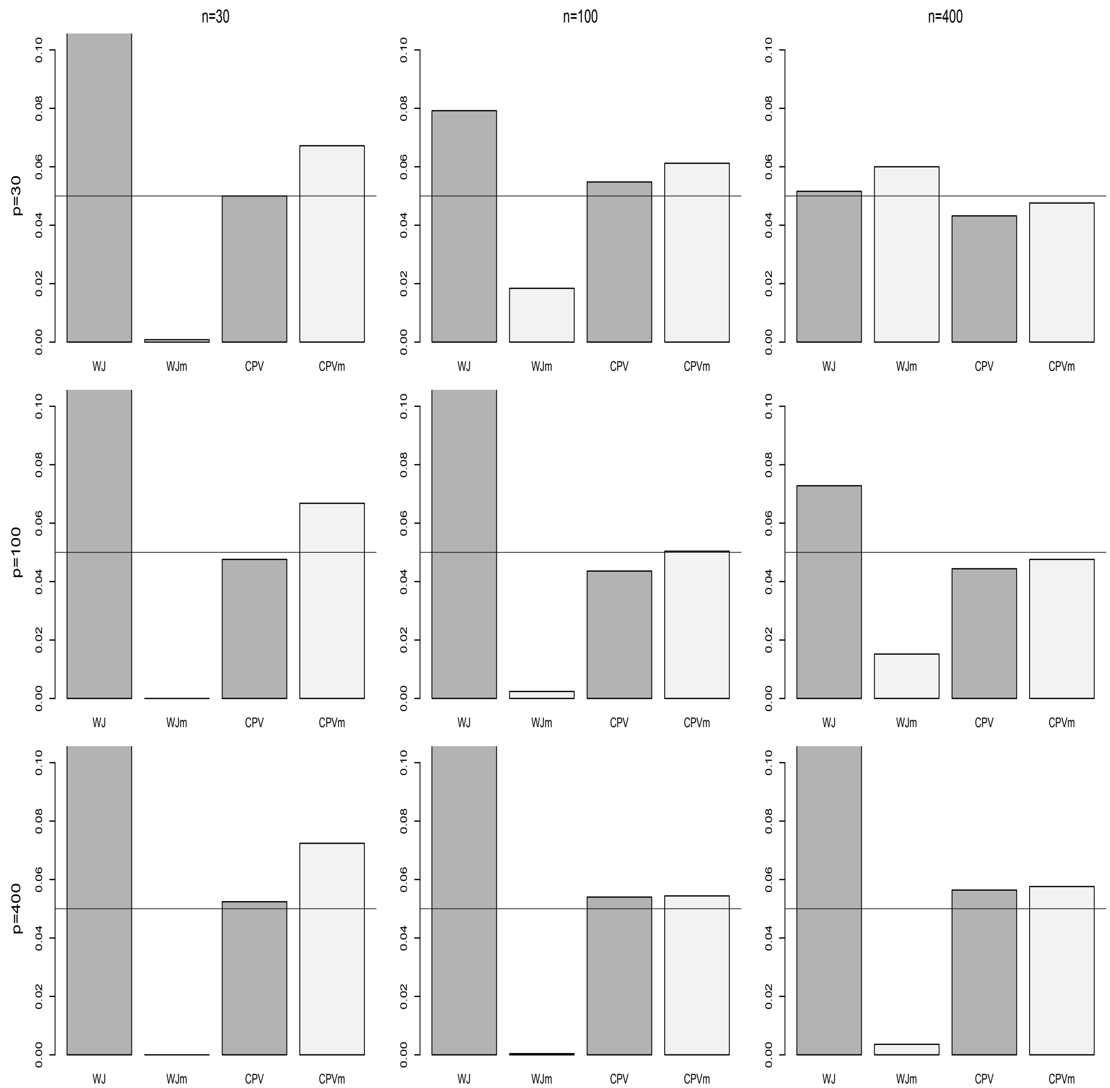

Fig. 3 Empirical null rejection frequencies, from 2,500 independent samples, of (i) the low-dimensional FvML test $\phi^{(n)}_{\mathrm{WJ}}$, (ii) its robustified version $\phi^{(n)}_{\mathrm{WJm}}$, (iii) the high-dimensional FvML test $\phi^{(n)}_{\mathrm{CPV}}$, and (iv) its robustified version $\phi^{(n)}_{\mathrm{CPVm}}$ (all performed at asymptotic level 5%), under p-dimensional FvML distributions for various dimensions p and sample sizes n; see Section 4.2 for details.

$$T^{(n)}_{\mathrm{WJm}} = \left(\frac{n^{1/2}(\|\bar{\mathbf{X}}_n\| - e_{10})}{\hat{\sigma}_n} \right)^2 \xrightarrow{\mathscr{L}} \chi^2_1 .$$

(ii) In view of the derivations above, the continuous mapping theorem implies that, for any $\boldsymbol{\theta} \in \mathscr{S}^{p-1}$ and $F \in \mathscr{F}_0$,

$$T^{(n)}_{\mathrm{WJm}} = \frac{n(\|\bar{\mathbf{X}}_n\| - e_{10})^2}{\mathrm{Var}[\mathbf{X}_1'\boldsymbol{\theta}]} + o_{\mathrm{P}}(1)$$

as $n \to \infty$ under $\mathscr{R}^{(n)}_p(\boldsymbol{\theta}, F)$. The result then follows from the fact that, under $\mathscr{R}^{(n)}_p(\boldsymbol{\theta}, F_{p,\kappa_0})$, with $\kappa_0 = h_p^{-1}(e_{10})$, $\mathrm{Var}[\mathbf{X}_1'\boldsymbol{\theta}] = 1 - \frac{p-1}{\kappa_0} e_{10} - e_{10}^2$; see, e.g., Lemma S.2.1 from [7]. $\square$

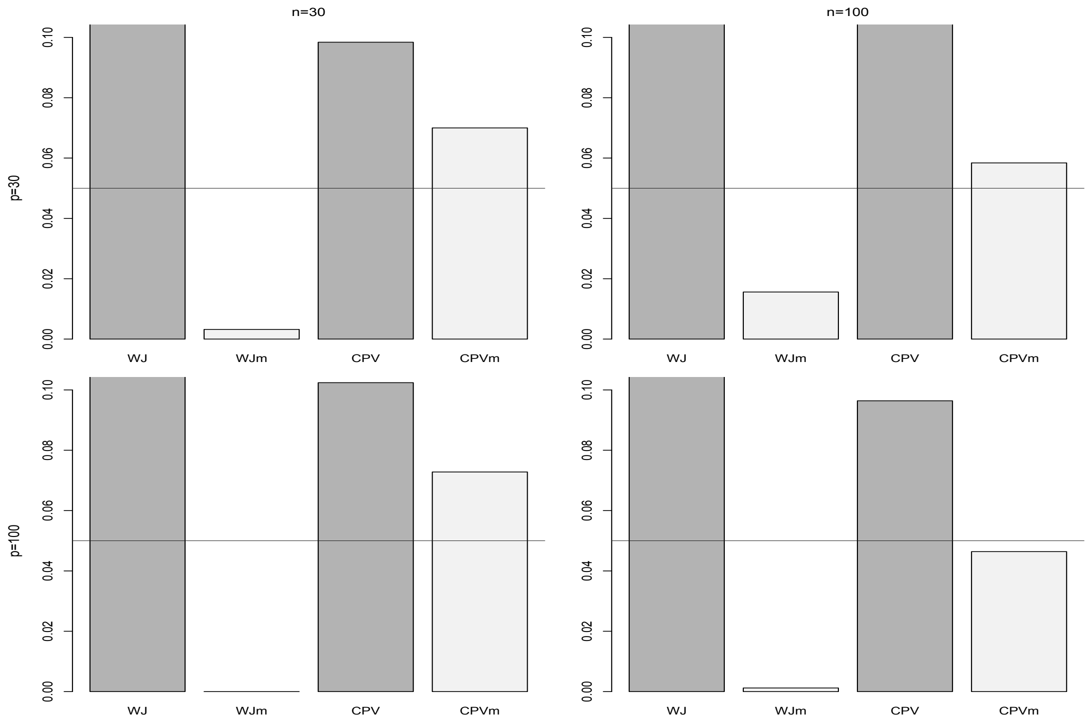

Fig. 4 Empirical null rejection frequencies, from 2,500 independent samples, of (i) the low-dimensional FvML test $\phi^{(n)}_{\rm WJ}$, (ii) its robustified version $\phi^{(n)}_{\rm WJm}$, (iii) the high-dimensional FvML test $\phi^{(n)}_{\rm CPV}$, and (iv) its robustified version $\phi^{(n)}_{\rm CPVm}$ (all performed at asymptotic level 5%), under p-dimensional Purkayastha distributions for various dimensions p and sample sizes n; see Section 4.2 for details.

Proof of Proposition 1. From Lemma S.2.1 in [7], we have that, under $\mathscr{R}^{(n)}_{p_n}(\boldsymbol{\theta}_n, F_{p_n,\kappa_n})$,

$$e_{n1} = \frac{I_{p_n/2}(\kappa_n)}{I_{p_n/2-1}(\kappa_n)} \ \text{ and } \ \tilde{e}_{n2} = 1 - \frac{p_n - 1}{\kappa_n}\, e_{n1} - e_{n1}^2.$$

The result then readily follows from

$$\frac{z}{\nu + 1 + \sqrt{z^2 + (\nu+1)^2}} \leq \frac{I_{\nu+1}(z)}{I_\nu(z)} \leq \frac{z}{\nu + \sqrt{z^2+\nu^2}} \tag{9}$$

for any $\nu, z > 0$; see (9) in [1]. □

Proof of Theorem 2. Writing $e_{n2} := \mathrm{E}[(\mathbf{X}'_{n1}\boldsymbol{\theta}_n)^2]$, Theorem 5.1 in [8] entails that, under $\mathscr{R}^{(n)}_{p_n}(\boldsymbol{\theta}_n, F_{p_n,\kappa_n})$, where (κ_n) is an *arbitrary* sequence in $(0,\infty)$,

$$\frac{\sqrt{p_n}\big(n\|\bar{\mathbf{X}}_n\|^2 - 1 - (n-1)e_{n1}^2\big)}{\sqrt{2}\,\big(p_n\tilde{e}_{n2}^2 + 2np_ne_{n1}^2\tilde{e}_{n2} + (1-e_{n2})^2\big)^{1/2}}$$

converges weakly to the standard normal distribution as $n \to \infty$. The result then follows from the fact that, under $\mathscr{R}^{(n)}_{p_n}(\boldsymbol{\theta}_n, F_{p_n,\kappa_n})$, where the sequence (κ_n) is such that, for any n, $e_{n1} = e_{10}$ under $\mathscr{R}^{(n)}_{p_n}(\boldsymbol{\theta}_n, F_{p_n,\kappa_n})$, one has

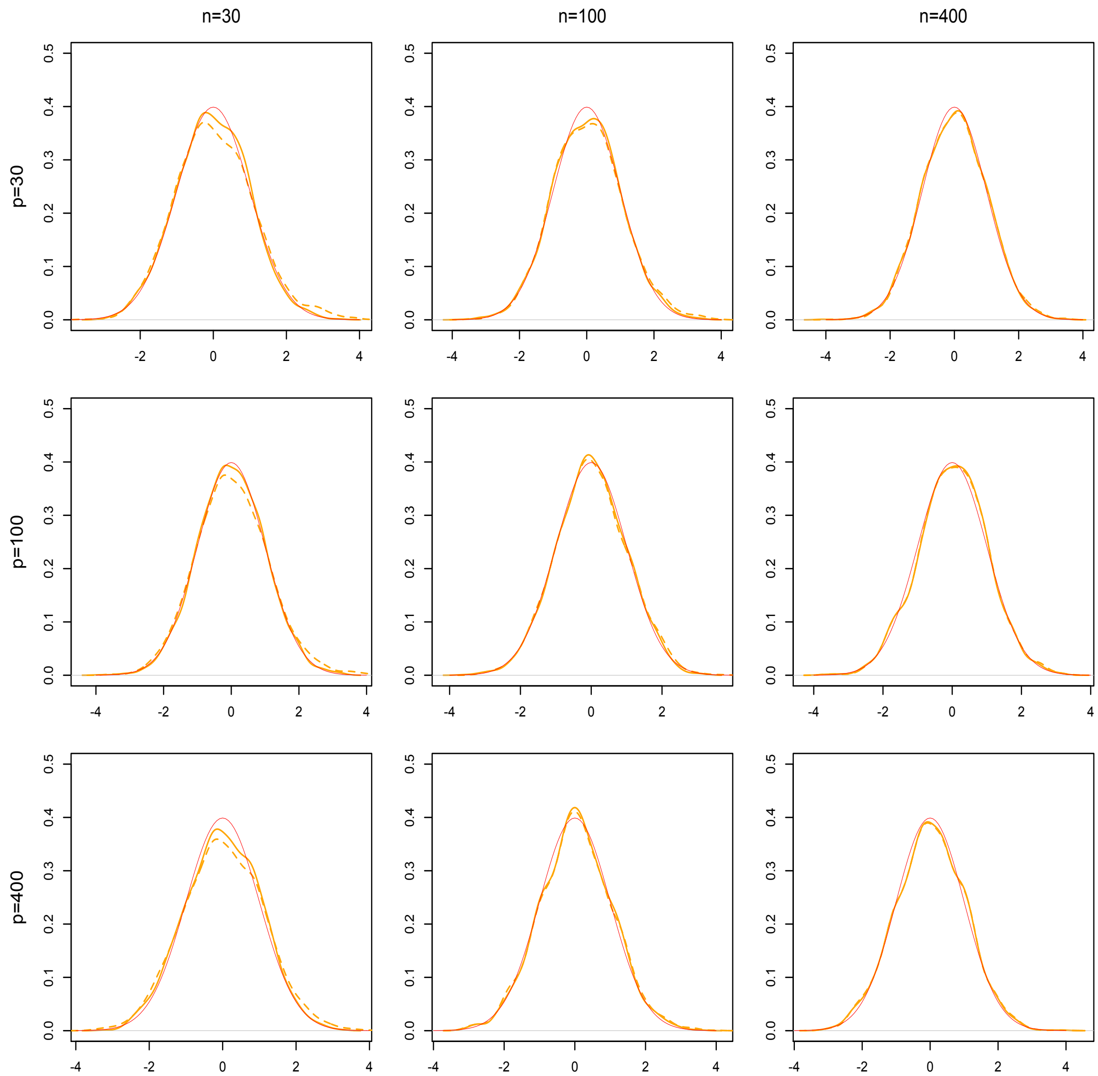

Fig. 5 Plots of kernel density estimators (based on Gaussian kernels and bandwidths resulting from the "rule of thumb" in [20]) of the (null) densities of $Q^{(n)}_{\rm CPV}$ (thick solid line) and $Q^{(n)}_{\rm CPVm}$ (thick dashed line) for various values of n and p, based on $M = 2,500$ random samples of size n from the p-dimensional FvML distribution with concentration $\kappa = p$; see Section 4.2 for details. For the sake of comparison, the standard normal density is also plotted (thin solid line).

$$e_{n2} = 1 - \frac{p_n - 1}{\kappa_n} e_{10}, \quad \tilde{e}_{n2} = 1 - \frac{p_n - 1}{\kappa_n} e_{10} - e_{10}^2, \quad \text{and} \quad \kappa_n/p_n \to c_0 \ \text{ as } n \to \infty;$$

see Proposition 1(ii). □

The proof of Theorem 3 requires the three following preliminary results.

Lemma 1. *Let Z be a random variable such that* $\mathrm{P}[|Z| \leq 1] = 1$. *Then* $\mathrm{Var}[Z^2] \leq 4\,\mathrm{Var}[Z]$.

Lemma 2. *Let the assumptions of Theorem 3 hold. Write $\hat{e}_{n1} = \|\bar{\mathbf{X}}_n\|$ and $\hat{e}_{n2} := \bar{\mathbf{X}}_n' \mathbf{S}_n \bar{\mathbf{X}}_n / \|\bar{\mathbf{X}}_n\|^2$. Then, as $n \to \infty$ under $\mathscr{R}^{(n)}_{p_n}(\boldsymbol{\theta}_n, F_{p_n,\kappa_n})$, (i) $(\hat{e}_{n1}^2 - e_{10}^2)/(e_{n2} - e_{10}^2) = o_{\mathrm{P}}(1)$ and (ii) $(\hat{e}_{2n} - e_{n2})/(e_{n2} - e_{10}^2) = o_{\mathrm{P}}(1)$.*

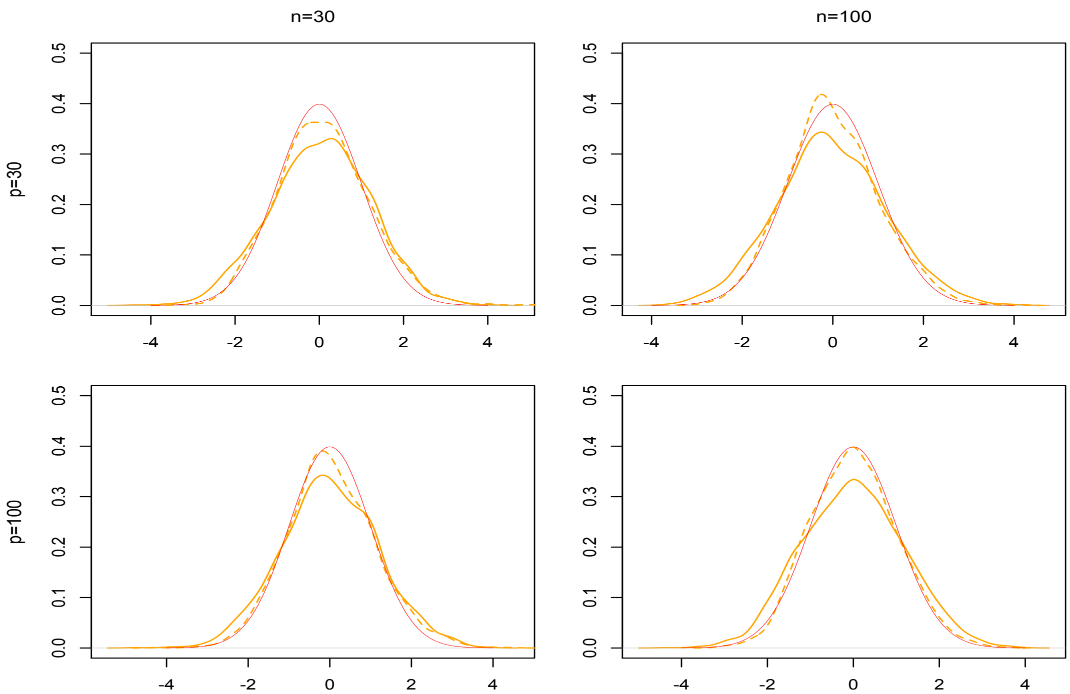

Fig. 6 Plots of kernel density estimators (based on Gaussian kernels and bandwidths resulting from the "rule of thumb" in [20]) of the (null) densities of $Q^{(n)}_{\rm CPV}$ (thick solid line) and $Q^{(n)}_{\rm CPVm}$ (thick dashed line) for various values of n and p, based on $M = 2{,}500$ random samples of size n from the p-dimensional Purkayastha distribution with concentration $\kappa = p$; see Section 4.2 for details. For the sake of comparison, the standard normal density is also plotted (thin solid line).

Lemma 3. *Let the assumptions of Theorem 3 hold. Write* $\sigma_n^2 := p_n(e_{n2} - e_{10}^2)^2 + 2np_n e_{10}^2(e_{n2} - e_{10}^2) + (1 - e_{n2})^2$ *and* $\hat{\sigma}_n^2 := p_n(\hat{e}_{n2} - \hat{e}_{n1}^2)^2 + 2np_n e_{10}^2(\hat{e}_{n2} - \hat{e}_{n1}^2) + (1 - \hat{e}_{n2})^2$*. Then* $(\hat{\sigma}_n^2 - \sigma_n^2)/\sigma_n^2 = o_{\rm P}(1)$ *as* $n \to \infty$ *under* $\mathscr{R}^{(n)}_{p_n}(\boldsymbol{\theta}_n, F_{p_n,\kappa_n})$.

Proof of Lemma 1. Let Z_a and Z_b be mutually independent and identically distributed with the same distribution as Z. Since $|x^2 - y^2| \leq 2|x - y|$ for any $x, y \in [-1, 1]$, we have that

$$\mathrm{Var}[Z^2] = \frac{1}{2}\,\mathrm{E}[(Z_a^2 - Z_b^2)^2] \leq 2\,\mathrm{E}[(Z_a - Z_b)^2] = 4\,\mathrm{Var}[Z],$$

which proves the result. □

Proof of Lemma 2. All expectations and variances in this proof are taken under the sequence of hypotheses $\mathscr{R}^{(n)}_{p_n}(\boldsymbol{\theta}_n, F_n)$ considered in the statement of Theorem 3, and all stochastic convergences are taken as $n \to \infty$ under the same sequence of hypotheses.

(i) Proposition 5.1 from [8] then yields

$$\mathrm{E}[\hat{e}_{n1}^2] = \mathrm{E}[\|\bar{\mathbf{X}}_n\|^2] = \frac{n-1}{n}\,e_{10}^2 + \frac{1}{n} \tag{10}$$

and

$$\mathrm{Var}[\hat{e}_{n1}^2] = \mathrm{Var}[\|\bar{\mathbf{X}}_n\|^2] = \frac{2(n-1)}{n^3}\,\tilde{e}_{2n}^2 + \frac{4(n-1)^2}{n^3}\,e_{10}^2\tilde{e}_{n2} + \frac{2(n-1)}{n^3(p_n-1)}(1-e_{n2}^2)^2 = \frac{4}{n}\,e_{10}^2\tilde{e}_{n2} + O(n^{-2}) \tag{11}$$

as $n\to\infty$. In view of Condition (i) in Theorem 3, this readily implies

$$\mathrm{E}\Big[\Big(\frac{\hat{e}_{n1}^2-e_{10}^2}{\tilde{e}_{n2}}\Big)^2\Big] = \mathrm{Var}\Big[\frac{\hat{e}_{n1}^2-e_{10}^2}{\tilde{e}_{n2}}\Big] + \Big(\mathrm{E}\Big[\frac{\hat{e}_{n1}^2-e_{10}^2}{\tilde{e}_{n2}}\Big]\Big)^2 = \frac{4e_{10}^2}{n\tilde{e}_{n2}} + O\Big(\frac{1}{n^2\tilde{e}_{n2}^2}\Big) + \Big(\frac{1-e_{10}^2}{n\tilde{e}_{n2}}\Big)^2 = o(1)$$

as $n\to\infty$, which establishes Part (i) of the result.

(ii) Write

$$\frac{\hat{e}_{n2}-e_{n2}}{\tilde{e}_{n2}} = \frac{1}{\tilde{e}_{n2}}\bigg(\Big(\frac{1}{\hat{e}_{n1}^2}-\frac{1}{e_{10}^2}\Big)\bar{\mathbf{X}}_n'\mathbf{S}_n\bar{\mathbf{X}}_n + \frac{1}{e_{10}^2}\,\bar{\mathbf{X}}_n'\mathbf{S}_n\bar{\mathbf{X}}_n - e_{n2}\bigg).$$

Part (i) of the result shows that $(\hat{e}_{n1}^2-e_{10}^2)/\tilde{e}_{n2}$ is $o_{\mathrm{P}}(1)$ as $n\to\infty$. Since (10)-(11) yield that $\hat{e}_{n1}$ converges in probability to $e_{10}(\neq 0)$, this implies that $(\hat{e}_{n1}^{-2}-e_{10}^{-2})/\tilde{e}_{n2}$ is $o_{\mathrm{P}}(1)$ as $n\to\infty$. This, and the fact that $\bar{\mathbf{X}}_n'\mathbf{S}_n\bar{\mathbf{X}}_n = O_{\mathrm{P}}(1)$ as $n\to\infty$, readily yields

$$\frac{\hat{e}_{n2}-e_{n2}}{\tilde{e}_{n2}} = \frac{1}{\tilde{e}_{n2}}\bigg(\frac{1}{e_{10}^2}\,\bar{\mathbf{X}}_n'\mathbf{S}_n\bar{\mathbf{X}}_n - e_{n2}\bigg) + o_{\mathrm{P}}(1) \tag{12}$$

as $n\to\infty$. Since

$$\frac{1}{e_{10}^2}\,\bar{\mathbf{X}}_n'\mathbf{S}_n\bar{\mathbf{X}}_n = \frac{1}{e_{10}^2}\,(\bar{\mathbf{X}}_n-e_{10}\boldsymbol{\theta})'\mathbf{S}_n(\bar{\mathbf{X}}_n-e_{10}\boldsymbol{\theta}) + \frac{2}{e_{10}}\,(\bar{\mathbf{X}}_n-e_{10}\boldsymbol{\theta})'\mathbf{S}_n\boldsymbol{\theta} + \boldsymbol{\theta}'\mathbf{S}_n\boldsymbol{\theta},$$

the result follows if we can prove that

$$A_n := \frac{1}{\tilde{e}_{n2}}(\bar{\mathbf{X}}_n-e_{10}\boldsymbol{\theta})'\mathbf{S}_n(\bar{\mathbf{X}}_n-e_{10}\boldsymbol{\theta}),\ \ B_n := \frac{1}{\tilde{e}_{n2}}(\bar{\mathbf{X}}_n-e_{10}\boldsymbol{\theta})'\mathbf{S}_n\boldsymbol{\theta},\quad \text{and}\quad C_n := \frac{1}{\tilde{e}_{n2}}(\boldsymbol{\theta}'\mathbf{S}_n\boldsymbol{\theta}-e_{n2})$$

all are $o_{\mathrm{P}}(1)$ as $n\to\infty$.

Starting with A_n, (10) yields

$$\mathrm{E}[|A_n|] \leq \frac{1}{\tilde{e}_{n2}}\,\mathrm{E}[\|\bar{\mathbf{X}}_n-e_{10}\boldsymbol{\theta}\|^2] = \frac{1}{\tilde{e}_{n2}}\Big(\frac{n-1}{n}\,e_{10}^2+\frac{1}{n}-e_{10}^2\Big) = \frac{1-e_{10}^2}{n\tilde{e}_{n2}} = o(1) \tag{13}$$

as $n\to\infty$. Since convergence in L_1 is stronger than convergence in probability, this implies that $A_n = o_{\mathrm{P}}(1)$ as $n\to\infty$. Turning to B_n, the Cauchy-Schwarz inequality and (13) provide

$$\mathrm{E}[|B_n|] \leq \frac{1}{\tilde{e}_{n2}}\,\mathrm{E}[\|\bar{\mathbf{X}}_n-e_{10}\boldsymbol{\theta}\|^2] = o(1),$$

as $n\to\infty$, so that B_n is indeed $o_{\mathrm{P}}(1)$ as $n\to\infty$. Finally, it follows from Lemma 1 that

$$\mathrm{E}[C_n^2] = \frac{1}{\tilde{e}_{n2}^2}\mathrm{E}[(\boldsymbol{\theta}'\mathbf{S}_n\boldsymbol{\theta}-e_{n2})^2] = \frac{1}{n\tilde{e}_{n2}^2}\mathrm{Var}[(\mathbf{X}_{n1}'\boldsymbol{\theta})^2] \leq \frac{4}{n\tilde{e}_{n2}} = o(1)$$

as $n\to\infty$, so that C_n is also $o_{\mathrm{P}}(1)$ as $n\to\infty$. This establishes the result. □

Proof of Lemma 3. As in the proof of Lemma 2, all expectations and variances in this proof are taken under the sequence of hypotheses $\mathscr{R}_{p_n}^{(n)}(\boldsymbol{\theta}_n,F_n)$ considered in the statement of Theorem 3, and all stochastic convergences are taken as $n\to\infty$ under the same sequence of hypotheses.

Let then $\tilde{\sigma}_n^2 := 2np_n e_{10}^2(e_{n2} - e_{10}^2)$. Since Condition (i) in Theorem 3 directly entails that $\sigma_n^2/\tilde{\sigma}_n^2 \to 1$ as $n \to \infty$, it is sufficient to show that $(\hat{\sigma}_n^2 - \sigma_n^2)/\tilde{\sigma}_n^2$ is $o_{\mathrm{P}}(1)$ as $n \to \infty$. To do so, write

$$\hat{\sigma}_n^2 - \sigma_n^2 = A_n + B_n + C_n, \tag{14}$$

where

$$A_n := p_n\left((\hat{e}_{n2} - \hat{e}_{n1}^2)^2 - (e_{n2} - e_{10}^2)^2\right), \quad B_n := 2np_n e_{10}^2\left(\hat{e}_{n2} - \hat{e}_{n1}^2 - e_{n2} + e_{10}^2\right),$$

and

$$C_n := (1 - \hat{e}_{n2})^2 - (1 - e_{n2})^2.$$

Since

$$\frac{|A_n|}{\tilde{\sigma}_n^2} \leq \frac{p_n}{\tilde{\sigma}_n^2} = \frac{1}{2ne_{10}^2(e_{n2} - e_{10}^2)} \quad \text{and} \quad \frac{|C_n|}{\tilde{\sigma}_n^2} \leq \frac{1}{\tilde{\sigma}_n^2} = \frac{1}{2np_n e_{10}^2(e_{n2} - e_{10}^2)},$$

almost surely, Condition (i) in Theorem 3 implies that $A_n/\tilde{\sigma}_n^2$ and $C_n/\tilde{\sigma}_n^2$ are $o_{\mathrm{P}}(1)$ as $n \to \infty$. The result then follows from the fact that, in view of Lemma 2,

$$\frac{B_n}{\tilde{\sigma}_n^2} = \frac{(\hat{e}_{n2} - e_{n2}) - (\hat{e}_{n1}^2 - e_{10}^2)}{e_{n2} - e_{10}^2}$$

is also $o_{\mathrm{P}}(1)$ as $n \to \infty$. □

Proof of Theorem 3. Decompose $Q_{\mathrm{CPVm}}^{(n)}$ into

$$Q_{\mathrm{CPVm}}^{(n)} = \frac{\sigma_n}{\hat{\sigma}_n} \times \frac{\sqrt{p_n}\left(n\|\bar{\mathbf{X}}_n\|^2 - 1 - (n-1)e_{10}^2\right)}{\sqrt{2}\,\sigma_n} =: \frac{\sigma_n}{\hat{\sigma}_n} \times V_n, \tag{15}$$

say. Theorem 5.1 in [8] entails that, under the sequence of hypotheses $\mathscr{R}_{p_n}^{(n)}(\boldsymbol{\theta}_n, F_n)$ considered in the statement of the theorem, V_n is asymptotically standard normal as $n \to \infty$. The result therefore follows from Lemma 3 and the Slutsky Lemma. □

References

1. Amos, D.E.: Computation of modified bessel functions and their ratios. Mathematics of Computation **28**(125), 239–251 (1974)
2. Banerjee, A., Ghosh, J.: Frequency sensitive competitive learning for scalable balanced clustering on high-dimensional hyperspheres. IEEE T. Neural Networ. **15**, 702–719 (2004)
3. Banerjee A., D.I.S.G.J., Sra, S.: Clustering on the unit hypersphere using von mises-fisher distributions. J. Mach. Learn. Res. **6**, 1345–1382 (2005)
4. Briggs, M.S.: Dipole and quadrupole tests of the isotropy of gamma-ray burst locations. The Astrophysical Journal **407**, 126–134 (1993)
5. Cai, T., Fan, J., Jiang, T.: Distributions of angles in random packing on spheres. J. Mach. Learn. Res. **14**, 1837–1864 (2013)
6. Cai, T., Jiang, T.: Phase transition in limiting distributions of coherence of high-dimensional random matrices. J. Multivariate anal. **107**, 24–39 (2012)
7. Cutting, C., Paindaveine, D., Verdebout, T.: Supplement to "testing uniformity on high-dimensional spheres against contiguous rotationally symmetric alternatives". Submitted (2015)

8. Cutting, C., Paindaveine, D., Verdebout, T.: Testing uniformity on high-dimensional spheres against contiguous rotationally symmetric alternatives. arXiv preprint arXiv:1502.02120 (2015)
9. Dryden, I.L.: Statistical analysis on high-dimensional spheres and shape spaces. Ann. Statist. **33**, 1643–1665 (2005)
10. Fisher, N.: Problems with the current definitions of the standard deviation of wind direction. Journal of Climate and Applied Meteorology **26**(11), 1522–1529 (1987)
11. Fisher, R.A.: Dispersion on a sphere. Proceedings of the Royal Society of London, ser. A **217**, 295–305 (1953)
12. Ko, D.: Robust estimation of the concentration parameter of the von mises-fisher distribution. The Annals of Statistics pp. 917–928 (1992)
13. Ko, D., Guttorp, P.: Robustness of estimators for directional data. The Annals of Statistics pp. 609–618 (1988)
14. Larsen, P., Blæsild, P., Sørensen, M.: Improved likelihood ratio tests on the von mises–fisher distribution. Biometrika **89**(4), 947–951 (2002)
15. Ley, C., Paindaveine, D., Verdebout, T.: High-dimensional tests for spherical location and spiked covariance. Journal of Multivariate Analysis, to appear (2015)
16. Ley, C., Verdebout, T.: Local powers of optimal one-and multi-sample tests for the concentration of fisher-von mises-langevin distributions. International Statistical Review **82**, 440–456 (2014)
17. Mardia, K.V., Jupp, P.E.: Directional statistics, vol. 494. Wiley. com (2009)
18. Paindaveine, D., Verdebout, T.: On high-dimensional sign tests. Bernoulli, to appear (2015)
19. Paindaveine, D., Verdebout, T.: Optimal rank-based tests for the location parameter of a rotationally symmetric distribution on the hypersphere. In M. Hallin, D. Mason, D. Pfeifer, and J. Steinebach Eds, Mathematical Statistics and Limit Theorems: Festschrift in Honor of Paul Deheuvels. Springer (to appear)
20. Silverman, B.W.: Density estimation for statistics and data analysis, vol. 26. CRC press (1986)
21. Stephens, M.: Multi-sample tests for the fisher distribution for directions. Biometrika **56**(1), 169–181 (1969)
22. Verdebout, T.: On some validity-robust tests for the homogeneity of concentrations on spheres. Journal of Nonparametric Statistics **27**, 372–383 (2015)
23. Watamori, Y., Jupp, P.E.: Improved likelihood ratio and score tests on concentration parameters of von mises–fisher distributions. Statistics & probability letters **72**(2), 93–102 (2005)
24. Watson, G.S.: Statistics on Spheres. Wiley, New York (1983)