

# On the Consistency of Incomplete U-statistics under Infinite Second-order Moments

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## Abstract

We derive a consistency result, in the  $L_1$ -sense, for incomplete U-statistics in the non-standard case where the kernel at hand has infinite second-order moments. Assuming that the kernel has finite moments of order  $p(\geq 1)$ , we obtain a bound on the  $L_1$  distance between the incomplete U-statistic and its Dirac weak limit, which allows us to obtain, for any fixed  $p$ , an upper bound on the consistency rate. Our results hold for most classical sampling schemes that are used to obtain incomplete U-statistics.

*Keywords:* consistency, convergence rates, incomplete U-statistics, moment assumptions

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## 1. Introduction

The concept of U-statistics, that was introduced in [Hoeffding \(1948\)](#), has met a tremendous success in probability and statistics. In this paper, we consider throughout random variables  $X_1, X_2, \dots$  that are independent copies of a random variable taking values in a generic space  $\mathcal{X}$  (that does not need to be  $\mathbb{R}^d$ ), and we fix a “kernel” function  $h : \mathcal{X}^\ell \rightarrow \mathbb{R}$  that is invariant under permutations of its arguments and that satisfies

$$\mathbb{E}[|h(X_1, \dots, X_\ell)|] < \infty \tag{1}$$

(throughout the paper,  $\ell$  is a fixed positive integer). The resulting U-statistic, namely

$$U_n := \frac{1}{\binom{n}{\ell}} \sum_{1 \leq i_1 < \dots < i_\ell \leq n} h(X_{i_1}, \dots, X_{i_\ell}), \tag{2}$$

is then an unbiased estimator for  $\theta := \mathbb{E}[h(X_1, \dots, X_\ell)]$ . Under these conditions,  $U_n$  converges almost surely to  $\theta$  as  $n \rightarrow \infty$  (see, e.g., Theorem A on page 190 from [Serfling \(1980\)](#)), which generalizes the classical strong law of large numbers that is obtained for  $\ell = 1$ . The asymptotic distribution theory for U-statistics was obtained under the second-order moment assumption stating that  $\mathbb{E}[h^2(X_1, \dots, X_\ell)] < \infty$ ; see, e.g., Section 5.5 from [Serfling \(1980\)](#) or Chapter 12 from [van der Vaart \(1998\)](#). In the standard case of “non-degenerate” U-statistics,  $\sqrt{n}(U_n - \theta)$  is asymptotically normal with mean zero, so that  $U_n - \theta = O_P(1/\sqrt{n})$ . For degenerate U-statistics, the consistency rate improves from  $1/\sqrt{n}$  to at least  $1/n$ .

Despite the many nice properties of U-statistics, practical applications are often jeopardized by the fact that it may be a heavy computational burden to evaluate the  $O(n^\ell)$  terms in  $U_n$ . This was the motivation to introduce the *incomplete U-statistics*

$$U_{n,N} := \frac{1}{N} \sum_{1 \leq i_1 < \dots < i_\ell \leq n} \alpha_n(i_1, \dots, i_\ell) h(X_{i_1}, \dots, X_{i_\ell}), \quad (3)$$

where the random variables  $\alpha_n(i_1, \dots, i_\ell)$ , with values in  $\{0, 1\}$ , take value one if and only if the term with index  $(i_1, \dots, i_\ell)$  was selected when drawing  $N$  terms without replacement out of the  $\binom{n}{\ell}$  terms in  $U_n$ . Incomplete U-statistics were first defined in [Blom \(1976\)](#) and were studied, among others, in [Brown and Kildea \(1978\)](#), [Enqvist \(1978\)](#), [Janson \(1984\)](#), [Lee \(1982\)](#), and [Nasari \(2012\)](#). Recent theoretical developments include, e.g., [Chen and Kato \(2019\)](#), which conducts an in-depth investigation of the asymptotic behaviour of incomplete U-statistics in a high-dimensional framework, and [Löwe and Terveer \(2021\)](#), where a central limit theorem is obtained for triangular arrays.

The asymptotic distribution theory for incomplete U-statistics was first derived in [Janson \(1984\)](#). However, like in subsequent papers, this was done there under the second-order moment assumption  $\mathbb{E}[h^2(X_1, \dots, X_\ell)] < \infty$ . To the best of our knowledge, a consistency result that would only require the milder first-order moment assumption in (1) is missing in the literature. In this note, we therefore establish such a consistency result (in the  $L_1$  sense) for the incomplete U-statistic in (3). Our results are actually more general: for any  $p \geq 1$ , we will obtain an upper bound on the  $L_1$  distance  $\mathbb{E}[|U_{n,N} - \theta|]$  under the  $p$ th-order moment assumption stating that  $\mathbb{E}[|h(X_1, \dots, X_\ell)|^p] < \infty$ . This will allow us to show consistency in the  $L_1$  sense and to derive an

upper bound on the corresponding consistency rate. While the asymptotic distribution theory of incomplete  $U$ -statistics was derived under the assumption that  $n/N \rightarrow 0$  as  $n \rightarrow \infty$ , we will make no restriction on the rate at which  $N$  diverges to infinity as  $n$  does. Actually, our upper bound on the  $L_1$  distance  $\mathbb{E}[|U_{n,N} - \theta|]$  even allows us to keep  $N$  fixed as  $n$  diverges to infinity.

The outline of the paper is as follows. In Section 2, we derive the aforementioned bound on  $\mathbb{E}[|U_{n,N} - \theta|]$  under the  $p$ th-order moment assumptions. In Section 3, we exploit the bound to establish consistency in the  $L_1$  sense and to obtain, for each fixed  $p$ , an upper bound on the corresponding consistency rate. In Section 4, we explain why our results actually extend to other classes of incomplete  $U$ -statistics. Finally, we perform a Monte-Carlo exercise in Section 5 to illustrate our consistency results.

## 2. A bound on the $L_1$ distance between $U_{n,N}$ and $\theta$

Before stating the main result of this section, we provide the following comments on the random variables  $\alpha(i_1, \dots, i_\ell)$  in (3), which will play an important role in our derivations. Under the considered sampling scheme, namely sampling without replacement, every fixed term from the complete  $U$ -statistic  $U_n$  in (2) is selected with probability

$$p_n := \mathbb{E}[\alpha_n(i_1, \dots, i_\ell)] = \frac{N}{\binom{n}{\ell}}.$$

Moreover, one has

$$\begin{aligned} & |\mathbb{E}[(\alpha_n(i_1, \dots, i_\ell) - p_n)(\alpha_n(j_1, \dots, j_\ell) - p_n)]| \\ & \leq \begin{cases} p_n & \text{if } (i_1, \dots, i_\ell) = (j_1, \dots, j_\ell) \\ p_n / \binom{n}{\ell} & \text{otherwise;} \end{cases} \quad (4) \end{aligned}$$

see the proof of Lemma 2.1 in [Janson \(1984\)](#). It is of course possible to provide exact values in (4), but our derivations below will only make use of the bounds in (4), which will allow us in Section 4 to extend in an effortless way our results to other ways to sample terms from  $U_n$ .

Under finite moment assumptions of order  $p$  (with  $p \in [1, \infty)$ ) on the kernel, we have the following bound on the  $L_1$  distance between the incomplete statistic  $U_{n,N}$  in (3) and its expectation  $\theta$ .

**Theorem 2.1.** *Let  $X_1, X_2, \dots$  be mutually independent and have a common distribution for which  $\mathbb{E}[|h(X_1, \dots, X_\ell)|^p] < \infty$  for some  $p \geq 1$ . Let  $(c_n)$  be an arbitrary sequence that diverges to infinity. Then,*

$$\mathbb{E}[|U_{n,N} - \theta|] = O\left(\frac{c_n^{(2-p)_+/2}}{\sqrt{\min(n, N)}}\right) + o\left(\frac{1}{c_n^{p-1}}\right) \quad (5)$$

as  $n \rightarrow \infty$ , with  $\theta = \mathbb{E}[h(X_1, \dots, X_\ell)]$  (throughout, we write  $x_+ := \max(x, 0)$ ).

The proof requires the following preliminary result, which we prove for the sake of completeness.

**Lemma 2.1.** *Let  $Z$  be a random variable taking values in  $\mathbb{R}^+$  and satisfying  $\mathbb{E}[Z^p] < \infty$  for some  $p \geq 1$ . Then,*

$$(i) \quad \mathbb{E}[Z\mathbb{I}[Z > c]] = o\left(\frac{1}{c^{p-1}}\right) \quad \text{and} \quad (ii) \quad \mathbb{E}[Z^2\mathbb{I}[Z \leq c]] = O(c^{(2-p)_+})$$

as  $c \rightarrow \infty$ .

*Proof of Lemma 2.1.* (i) Since the result trivially follows from Lebesgue's Dominated Convergence Theorem for  $p = 1$ , we may assume that  $p > 1$ . Letting  $q = p/(p-1)$  be the conjugate exponent of  $p$ , Hölder's inequality yields  $\mathbb{E}[Z\mathbb{I}[Z > c]] \leq (\mathbb{E}[Z^p])^{1/p} (P[X > c])^{1/q} = (\mathbb{E}[Z^p])^{1/p} (c^p P[X > c])^{1/q} c^{-p/q}$ . Since  $p/q = p-1$ , the result follows from the fact that the moment assumption on  $Z$  ensures that  $c^p P[X > c] = o(1)$  as  $c \rightarrow \infty$  (see, e.g., the corollary on page 47 of [Serfling \(1980\)](#)). (ii) Let us assume that  $p \in [1, 2)$  (the result is trivial for  $p \geq 2$ ). Then,  $\mathbb{E}[Z^2\mathbb{I}[Z \leq c]] \leq \mathbb{E}[c^{2-p} Z^p \mathbb{I}[Z \leq c]] \leq c^{2-p} \mathbb{E}[Z^p] = O(c^{2-p})$  as  $c \rightarrow \infty$ , which establishes the result.  $\square$

We can now prove [Theorem 2.1](#).

*Proof of Theorem 2.1.* In this proof,  $\mathcal{I}_{n,\ell} = \{I = (i_1, \dots, i_\ell) : 1 \leq i_1 < \dots < i_\ell \leq n\}$  will denote the collection of possible multi-indices  $I = (i_1, \dots, i_\ell)$  in the complete U-statistic  $U_n$ . This allows us to write

$$U_n = \frac{1}{\binom{n}{\ell}} \sum_{I \in \mathcal{I}_{n,\ell}} h(X_I) \quad \text{and} \quad U_{n,N} = \frac{1}{N} \sum_{I \in \mathcal{I}_{n,\ell}} \alpha_n(I) h(X_I),$$

where  $X_I$  obviously stands for  $(X_{i_1}, \dots, X_{i_\ell})$ . Based on the truncated kernel defined by

$$h_n(x_1, \dots, x_\ell) := h(x_1, \dots, x_\ell) \mathbb{I}[|h(x_1, \dots, x_\ell)| \leq c_n],$$

define then  $\theta_n := \mathbb{E}[h_n(X_1, \dots, X_\ell)]$ ,

$$\tilde{U}_{n,N} := \frac{1}{N} \sum_{I \in \mathcal{I}_{n,\ell}} \alpha_n(I) h_n(X_I)$$

and

$$\bar{U}_{n,N} := \frac{1}{N} \sum_{I \in \mathcal{I}_{n,\ell}} \alpha_n(I) \mathbb{E}[h_n(X_I)] = \frac{\theta_n}{N} \sum_{I \in \mathcal{I}_{n,\ell}} \alpha_n(I).$$

With this notation, write

$$\begin{aligned} \mathbb{E}[|U_{n,N} - \theta|] &\leq \mathbb{E}[|U_{n,N} - \tilde{U}_{n,N}|] + \mathbb{E}[|\tilde{U}_{n,N} - \bar{U}_{n,N}|] + \mathbb{E}[|\bar{U}_{n,N} - \theta_n|] + |\theta_n - \theta| \\ &=: T_{1n} + T_{2n} + T_{3n} + T_{4n}, \end{aligned}$$

say. We establish the result by proving that, for any  $j = 1, 2, 3, 4$ ,

$$T_{jn} = O\left(\frac{c_n^{(2-p)+/2}}{\sqrt{\min(n, N)}}\right) + o\left(\frac{1}{c_n^{p-1}}\right) \quad (6)$$

(throughout the proof, all  $o$ 's and  $O$ 's are as  $n \rightarrow \infty$ ). In view of the moment assumption on  $h(X_1, \dots, X_\ell)$ , Lemma 2.1(i) yields that

$$\begin{aligned} T_{4n} = |\theta_n - \theta| &\leq \mathbb{E}[|h_n(X_1, \dots, X_\ell) - h(X_1, \dots, X_\ell)|] \\ &= \mathbb{E}[|h(X_1, \dots, X_\ell)| \mathbb{I}[|h(X_1, \dots, X_\ell)| > c_n]] \\ &= o\left(\frac{1}{c_n^{p-1}}\right), \end{aligned}$$

so that  $T_{4n}$  satisfies (6). By construction,

$$\bar{U}_{n,N} = \frac{\theta_n}{N} \sum_{I \in \mathcal{I}_{n,\ell}} \alpha_n(I) = \theta_n, \quad (7)$$

so that  $T_{3n}$  trivially satisfies (6), too (the motivation to consider  $T_{3n}$  will be made clear when extending the results to other sampling schemes in Section 4). Now,

$$\begin{aligned}
T_{1n} &= \mathbb{E}[|U_{n,N} - \tilde{U}_{n,N}|] \\
&\leq \frac{1}{N} \sum_{I \in \mathcal{I}_{n,\ell}} \mathbb{E}[\alpha_n(I)] \mathbb{E}[|h_n(X_I) - h(X_I)|] \\
&= \mathbb{E}[|h(X_1, \dots, X_\ell)| \mathbb{I}[|h(X_1, \dots, X_\ell)| > c_n]] \\
&= o\left(\frac{1}{c_n^{p-1}}\right),
\end{aligned}$$

so that  $T_{1n}$  satisfies (6) as well. It remains to treat  $T_{2n}$ , which is more delicate. First note that

$$\tilde{U}_{n,N} - \bar{U}_{n,N} = \frac{1}{N} \sum_{I \in \mathcal{I}_{n,\ell}} \alpha_n(I) g_n(X_I),$$

where  $g_n$  is the centered truncated kernel defined by

$$g_n(x_1, \dots, x_\ell) := h_n(x_1, \dots, x_\ell) - \mathbb{E}[h_n(X_1, \dots, X_\ell)].$$

Since  $g_n$  is bounded for any  $n$  by construction, we can consider

$$\begin{aligned}
T_{2n}^2 &\leq \mathbb{E}[(\tilde{U}_{n,N} - \bar{U}_{n,N})^2] = \frac{1}{N^2} \sum_{I, J \in \mathcal{I}_{n,\ell}} \mathbb{E}[\alpha_n(I) \alpha_n(J) g_n(X_I) g_n(X_J)] \\
&= \frac{1}{N^2} \sum_{I, J \in \mathcal{I}_{n,\ell}} \mathbb{E}[\alpha_n(I) \alpha_n(J)] \mathbb{E}[g_n(X_I) g_n(X_J)] \\
&\leq a_n + b_n,
\end{aligned}$$

with (recall (4))

$$a_n := \frac{p_n}{N^2} \sum_{I \in \mathcal{I}_{n,\ell}} \mathbb{E}[g_n^2(X_I)] \quad \text{and} \quad b_n := \frac{2p_n^2}{N^2} \sum_{I, J \in \mathcal{I}_{n,\ell}, I \neq J} \mathbb{E}[g_n(X_I) g_n(X_J)];$$

for  $b_n$ , we used (4) to obtain

$$\mathbb{E}[\alpha_n(I) \alpha_n(J)] = \text{Cov}[\alpha_n(I), \alpha_n(J)] + p_n^2 \leq \frac{p_n}{\binom{n}{\ell}} + p_n^2 = \frac{p_n^2}{N} + p_n^2 \leq 2p_n^2. \quad (8)$$

For  $a_n$ , Lemma 2.1(ii) yields

$$\begin{aligned}
a_n &= \frac{p_n}{N^2} \binom{n}{\ell} \mathbb{E}[g_n^2(X_1, \dots, X_\ell)] = \frac{1}{N} \text{Var}[h_n(X_1, \dots, X_\ell)] \\
&\leq \frac{1}{N} \mathbb{E}[h_n^2(X_1, \dots, X_\ell)] = \frac{1}{N} \mathbb{E}[h^2(X_1, \dots, X_\ell) \mathbb{I}[|h(X_1, \dots, X_\ell)| \leq c_n]] \\
&= O\left(\frac{c_n^{(2-p)_+}}{\min(n, N)}\right). \tag{9}
\end{aligned}$$

Consider then  $b_n$ . If the multi-indices  $I$  and  $J$  do not have at least one index in common, then mutual independence of the  $X_i$ 's readily yields

$$\mathbb{E}[g_n(X_I)g_n(X_J)] = \mathbb{E}[g_n(X_I)]\mathbb{E}[g_n(X_J)] = 0.$$

Out of the

$$\binom{n}{\ell} \left( \binom{n}{\ell} - 1 \right)$$

terms in  $b_n$ , we may thus restrict to the

$$m_n := \binom{n}{\ell} \left( \binom{n}{\ell} - 1 \right) - \binom{n}{\ell} \binom{n-\ell}{\ell}$$

terms for which  $I$  and  $J$  have at least one index in common. The Cauchy–Schwarz inequality and Lemma 2.1(ii) then provide

$$\begin{aligned}
b_n &\leq \frac{2m_n p_n^2}{N^2} \mathbb{E}[g_n^2(X_1, \dots, X_\ell)] = \frac{2m_n}{\binom{n}{\ell}^2} \text{Var}[h_n(X_1, \dots, X_\ell)] \\
&\leq \frac{2m_n}{\binom{n}{\ell}^2} \mathbb{E}[h_n^2(X_1, \dots, X_\ell)] \leq \frac{2m_n}{\binom{n}{\ell}^2} \mathbb{E}[h^2(X_1, \dots, X_\ell) \mathbb{I}[|h(X_1, \dots, X_\ell)| \leq c_n]] \\
&= \frac{2m_n}{\binom{n}{\ell}^2} O(c_n^{(2-p)_+}) = O\left(\frac{c_n^{(2-p)_+}}{\min(n, N)}\right), \tag{10}
\end{aligned}$$

where we used the fact that

$$\begin{aligned}
\frac{m_n}{\binom{n}{\ell}^2} &= 1 - \frac{1}{\binom{n}{\ell}} - \frac{\binom{n-\ell}{\ell}}{\binom{n}{\ell}} \\
&= 1 - \frac{(n-\ell)(n-\ell-1)\dots(n-2\ell+1)}{n(n-1)\dots(n-\ell+1)} + O\left(\frac{1}{n}\right) \\
&= 1 - \left(1 - \frac{\ell}{n}\right)\left(1 - \frac{\ell}{n-1}\right)\dots\left(1 - \frac{\ell}{n-\ell+1}\right) + O\left(\frac{1}{n}\right) \\
&= O\left(\frac{1}{n}\right).
\end{aligned}$$

Thus, (9)–(10) yield

$$T_{2n} = O\left(\frac{c_n^{(2-p)_+/2}}{\sqrt{\min(n, N)}}\right).$$

This shows that  $T_{2n}$  satisfies (6), too, which establishes the result.  $\square$

Under the minimal assumption making  $\theta$  well-defined, namely under the assumption that  $E[|h(X_1, \dots, X_\ell)|] < \infty$  ( $p = 1$ ), a direct corollary of Theorem 2.1 is that  $E[|U_{n,N} - \theta|] = o(1)$  as soon as  $\min(n, N) \rightarrow \infty$  (this follows by taking, e.g.,  $c_n = \log(\min(n, N))$ ). This establishes convergence in the  $L_1$  sense (hence also in probability) of  $U_{n,N}$  to its expectation  $\theta$ . Interestingly, convergence holds even when  $N$  would diverge to  $n$  arbitrarily slowly as a function of  $n$ , whereas classical asymptotic theory of incomplete U-statistics requires that  $n/N \rightarrow 0$  as  $n \rightarrow \infty$ ; see Janson (1984).

### 3. Consistency rates

As just pointed out, first-order moment assumptions on  $h(X_1, \dots, X_\ell)$  are sufficient to ensure convergence of incomplete U-statistics to their expectation. Actually, the bound obtained in Theorem 2.1 also allows us to comment on the convergence rate that is achieved when assuming finite moments of order  $p(\geq 1)$ .

Consider first the case  $p \geq 2$ . Taking  $c_n = \sqrt{\min(n, N)}$  in (5) then provides

$$E[|U_{n,N} - \theta|] = O\left(\frac{1}{\sqrt{\min(n, N)}}\right), \quad (11)$$



which is compatible with the (optimal) root- $n$  consistency rate that was obtained in [Janson \(1984\)](#) under second-order moment assumptions ([Janson \(1984\)](#) restricts to the case where  $n/N = o(1)$ , in which case the optimal rate is indeed  $1/\sqrt{n}$ , but it should be clear that the optimal rate generalizes into the one in (11) when no assumption is made on the rate at which  $N$  diverges to infinity as  $n$  does). Now, consider the case  $p \in [1, 2)$ , that is the one of main interest in the present work. Would both terms in the righthand side of (5) be big  $O$ 's, an upper bound on the consistency rate could be obtained by choosing  $c_n$  so that

$$\frac{c_n^{(2-p)_+/2}}{\sqrt{\min(n, N)}} \sim \frac{1}{c_n^{p-1}}$$

(where  $a_n \sim b_n$  means that  $a_n = O(b_n)$  and  $b_n = O(a_n)$ ), that is, by taking  $c_n \sim \{\min(n, N)\}^{1/p}$ , which would provide

$$\mathbb{E}[|U_{n,N} - \theta|] = O\left(\frac{1}{\{\min(n, N)\}^{(p-1)/p}}\right).$$

Since the second term in the righthand side of (5) is a little  $o$  rather than a big  $O$ , it is possible to take  $c_n = r_n \{\min(n, N)\}^{1/p}$ , with a suitable sequence  $(r_n)$  that is  $o(1)$ , to obtain

$$\mathbb{E}[|U_{n,N} - \theta|] = o\left(\frac{1}{\{\min(n, N)\}^{(p-1)/p}}\right) \quad (12)$$

for  $p \in [1, 2)$ . This is only an upper bound on the convergence rate in the  $L_1$  sense, but the natural values of this upper bound for  $p = 1$  and  $p \nearrow 2$ , namely

$$\mathbb{E}[|U_{n,N} - \theta|] = o(1) \quad \text{and} \quad \mathbb{E}[|U_{n,N} - \theta|] = o\left(\frac{1}{\sqrt{\min(n, N)}}\right),$$

respectively, lead us to conjecture that this is actually the exact convergence rate. Note also that, as expected, the larger  $p \in [1, 2)$ , the better the convergence rate in (12).

#### 4. Extensions to other sampling schemes

The incomplete U-statistic  $U_{n,N}$  considered in the earlier sections is the one in (3), that is obtained by *sampling without replacement*  $N$  terms from

the  $\binom{n}{\ell}$  terms in the complete U-statistic  $U_n$  in (2). Other classical sampling schemes are possible, including both following ones:

- *Sampling with replacement*: in this scheme, the  $\binom{n}{\ell}$  random variables  $\alpha(i_1, \dots, i_\ell)$ , with values in  $\{0, 1, \dots, N\}$ , are the marginals of a multinomial random vector with count parameter  $N$  and homogeneous probabilities  $1/\binom{n}{\ell}, \dots, 1/\binom{n}{\ell}$  on each of the  $\binom{n}{\ell}$  possible outcomes. Here,  $N$  is an arbitrary positive integer and we let  $p_n = N/\binom{n}{\ell}$ .
- *Bernoulli sampling*: in this scheme, the  $\binom{n}{\ell}$  random variables  $\alpha(i_1, \dots, i_\ell)$  form a random sample from the Bernoulli distribution with success probability  $p_n = N/\binom{n}{\ell}$ , with  $N \in (0, \binom{n}{\ell}]$ .

For *sampling with replacement*, it is easy to check that, using the notation from the proof of Theorem 2.1, we still have

$$N \geq 1, \tag{13}$$

$$\sum_{I \in \mathcal{I}_{n,\ell}} \alpha_n(I) = N, \tag{14}$$

$$\mathbb{E}[\alpha_n(I)] = p_n = \frac{N}{\binom{n}{\ell}} \tag{15}$$

and that the covariance bounds in (4) still hold. Since these were the only assumptions used on  $N$  and on the random variables  $\alpha(i_1, \dots, i_\ell)$  in the proof of Theorem 2.1, we conclude that this result, hence also its consequences described at the end of Section 2 and in Section 3, extend to incomplete U-statistics that are obtained from *sampling with replacement*.

The situation is slightly more complicated for *Bernoulli sampling*, for which (15) and the covariance bounds in (4) still hold, but for which (13)–(14) do not hold in general. An inspection of the proof of Theorem 2.1 reveals that the proof can easily be adapted to the case where  $N$  remains away from zero — which is obviously an extremely mild assumption since consistency requires anyway that  $N$  diverges to infinity as  $n$  does. The fact that (14) that does not hold for *Bernoulli sampling* is more serious, since it implies that (7) does not hold for this sampling scheme, which forces us to control  $T_{3n}$ : since

$$\bar{U}_{n,N} - \theta_n = \frac{\theta_n}{N} \sum_{I \in \mathcal{I}_{n,\ell}} (\alpha_n(I) - p_n),$$

using (4) and the fact that  $\theta_n = \theta + o(1)$  in the proof of Theorem 2.1 yields

$$\begin{aligned} T_{3n}^2 &\leq \mathbb{E}[(\bar{U}_{n,N} - \theta_n)^2] = \frac{\theta_n^2}{N^2} \sum_{I, J \in \mathcal{I}_{n,\ell}} \mathbb{E}[(\alpha_n(I) - p_n)(\alpha_n(J) - p_n)] \\ &\leq \frac{\theta_n^2}{N^2} \left\{ \binom{n}{\ell} \times p_n + \binom{n}{\ell} \left( \binom{n}{\ell} - 1 \right) \times \frac{p_n}{\binom{n}{\ell}} \right\} \\ &\leq \frac{2\theta_n^2 \binom{n}{\ell} p_n}{N^2} = \frac{2\theta_n^2}{N} = O\left(\frac{1}{N}\right) \end{aligned}$$

as  $n$  diverges to infinity. Thus,

$$T_{3n} = O\left(\frac{1}{\sqrt{N}}\right) = O\left(\frac{c_n^{(2-p)+/2}}{\sqrt{\min(n, N)}}\right)$$

as  $n$  diverges to infinity, which shows that  $T_{3n}$  satisfies (6) for *Bernoulli sampling*, too. Therefore, we conclude that all consistency results of the paper apply to this third sampling scheme as well.

## 5. A numerical illustration

We end this paper with a Monte-Carlo exercise we performed in order to illustrate our consistency results. In this numerical exercise, we focus on scalar observations ( $\mathcal{X} = \mathbb{R}$ ) and on the kernel  $h : \mathbb{R}^2 \rightarrow \mathbb{R}$  defined by

$$h(x_1, x_2) = |x_1 - x_2|.$$

For each  $n \in \{50, 100, 150, \dots, 400\}$ , we generated  $M = 12,000$  random samples of size  $n$  from the Student  $t$  distribution with  $\nu = 1.5$  (for which the kernel  $h(X_1, X_2)$  has infinite second-order moments). In each of the resulting samples, we evaluated the complete U-statistic

$$U_n = \frac{1}{\binom{n}{2}} \sum_{1 \leq i < j \leq n} |X_i - X_j| \quad (16)$$

and nine incomplete U-statistics of the form

$$U_{n,N} = \frac{1}{N} \sum_{1 \leq i < j \leq n} \alpha(i, j) |X_i - X_j|, \quad (17)$$

obtained by combining one of the three possible sampling schemes (*sampling without replacement*, *sampling with replacement*, and *Bernoulli sampling*) and a value of  $N \in \{n^{3/2}, n, n^{2/3}\}$  (recall that the complete statistic involves  $\binom{n}{2} = n(n-1)/2$  terms). Complete and incomplete U-statistics here are consistent for  $\theta = \mathbb{E}[|X_1 - X_2|]$ . For each value of  $n$  and for each given complete or incomplete U-statistic,  $V_n$  say, we evaluated the empirical  $L_1$  distance

$$\frac{1}{M} \sum_{m=1}^M |V_n(m) - \theta| \quad (18)$$

between  $V_n$  and  $\theta$ , where  $V_n(m)$  denotes the value taken by  $V_n$  in the  $m$ th random sample of size  $n$  that was generated above. In Figure 1, we plot the quantities in (18) versus the sample size  $n$  for the complete U-statistic and each of the nine incomplete statistics mentioned above. To explore the sensitivity of the results to moment assumptions on  $h(X_1, X_2)$ , we repeated this exercise for random samples generated from Student  $t$  distributions with  $\nu = 1.8$  and 2.1 degrees of freedom, then from the standard Gaussian distribution (note that, among the four distributions considered, the kernel  $h(X_1, X_2)$  has infinite second-order moments for the Student  $t$  distributions with  $\nu = 1.5$  and  $\nu = 1.8$  degrees of freedom only).

Figure 1 clearly supports our theoretical results. Irrespective of moment assumptions, empirical  $L_1$  distances of all complete and incomplete statistics decrease to zero as  $n$  increases (the kink at  $n = 250$  in the curves associated with  $\nu = 1.5$  results from the heavy tailed distribution of  $V_n(m)$  in that case, which significantly slows down the convergence of the Monte Carlo simulation). As expected, the smaller  $N$ , the larger these distances, but the results also reveal that, irrespective of the adopted sampling scheme,  $N = n^{3/2}$  provide virtually the same  $L_1$  distances as the complete U-statistics. Maybe less importantly, this numerical exercise also shows that, while sampling with replacement and sampling without replacement provide the same performances, Bernoulli sampling yields higher  $L_1$  distances, with a difference that increases as  $N$  decreases.

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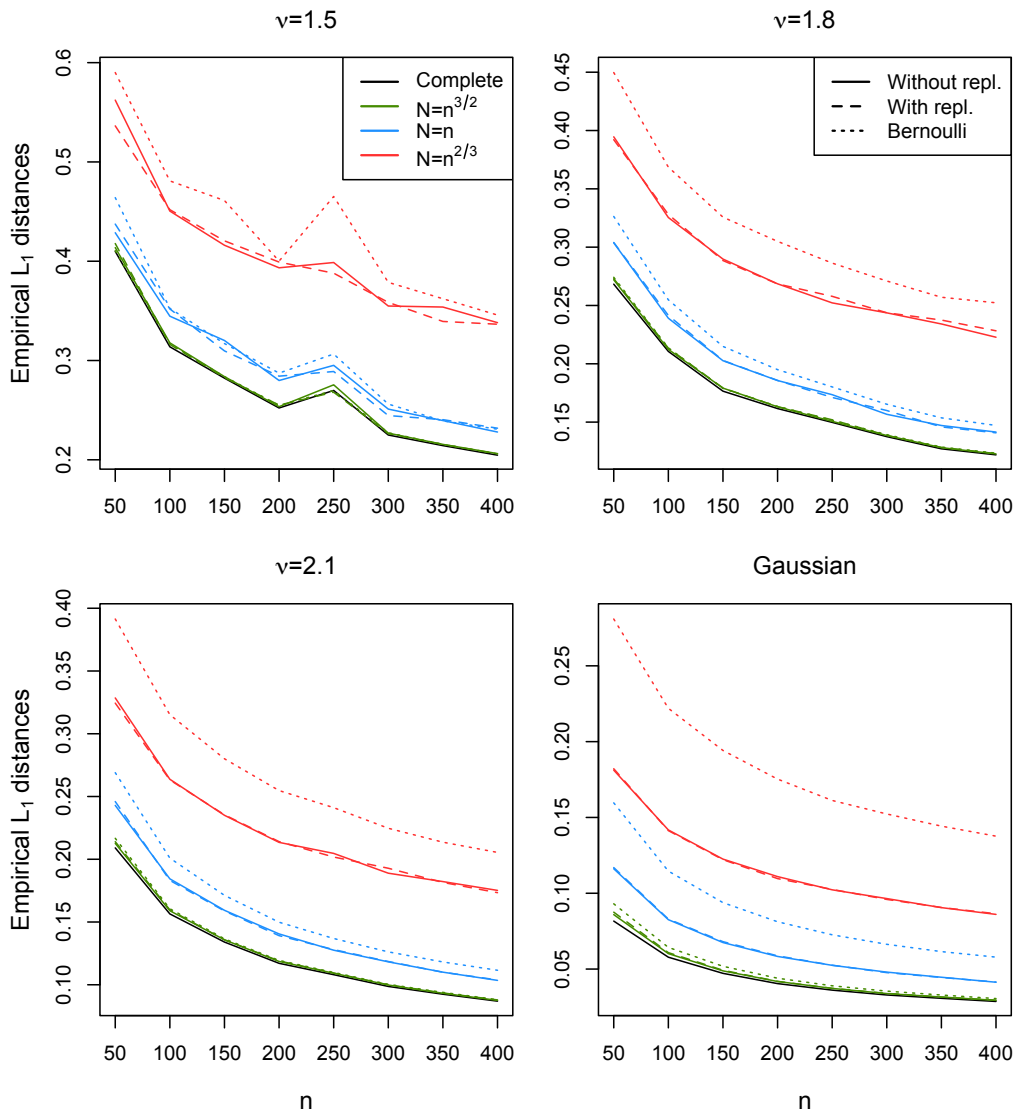


Figure 1: Empirical  $L_1$  distances in (18) versus the sample size  $n$  for the complete U-statistic in (16) and the nine versions of the incomplete U-statistic in (16), when observations are randomly sampled from the Student  $t$  distribution with  $\nu = 1.5$  degrees of freedom (top left),  $\nu = 1.8$  degrees of freedom (top right),  $\nu = 2.1$  degrees of freedom (bottom left), or from the standard Gaussian distribution (bottom right). The nine versions of the incomplete U-statistic result from the combination of one of the three possible sampling schemes (*sampling without replacement*, *sampling with replacement*, and *Bernoulli sampling*) and one of the three considered values of  $N \in \{n^{3/2}, n, n^{2/3}\}$ .