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On UMPS Hypothesis Testing

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Abstract For two-sided hypothesis testing in location families, the classical optimality criterion is the one leading to *uniformly most powerful unbiased (UMPU)* tests. Such optimal tests, however, are constructed in exponential models only. We argue that if the base distribution is symmetric, then it is natural to consider *uniformly most powerful symmetric (UMPS)* tests, that is, tests that are uniformly most powerful in the class of level-α tests whose power function is symmetric. For single-observation models, we provide a condition ensuring existence of UMPS tests and provide their explicit form. When this condition is not met, UMPS tests may fail to exist and we provide a weaker condition under which there exist UMP tests in the class of level-α tests whose power function is symmetric and U-shaped. In the multi-observation case, we obtain results in exponential models that also allow for non-location families.

Keywords Exponential families · Hypothesis testing · Statistical principle · UMP tests · UMPU tests

1 Introduction

Let $(\mathcal{X}, \mathcal{A}, \mathcal{P} = \{\mathrm{P}_\theta : \theta \in \Theta \subset \mathbb{R}\})$ be a parametric statistical model indexed by a scalar parameter θ. We consider the problem of testing the null hypothesis $\mathcal{H}_0 : \theta = \theta_0$ against the alternative hypothesis $\mathcal{H}_1 : \theta \neq \theta_0$ at level α, where θ_0 is a fixed parameter value. While the corresponding one-sided testing problems allow for *uniformly most powerful (UMP)* tests under the mild *monotone likelihood ratio* assumption (see, e.g., Section 3.4 in Lehmann and Romano (2022) or Section 6.1.2 in Shao (2003)), no UMP tests do exist for two-sided problems. Classically, the issue is solved by resorting to the *unbiasedness*

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principle, which consists in restricting to tests that are unbiased at level α, that is, to level-α tests ϕ such that $\mathrm{E}_\theta[\phi] \geq \alpha$ for any $\theta \neq \theta_0$. Existence of *uniformly most powerful unbiased (UMPU)* tests, however, is guaranteed only in exponential families (see, e.g., Section 4.2 in Lehmann and Romano (2022) or Section 6.2.2 in Shao (2003)).

In this work, we show that there are cases where the unbiasedness principle may be replaced most naturally with a *symmetry principle*, that consists in restricting to level-α tests whose power function is symmetric about θ_0. A prototypical example of this nature is the one where we observe a random sample $X = (X_1, \ldots, X_n)$ from the density

$$f_\theta(x) = f_0(x - \theta), \quad x \in \mathbb{R},$$

where $\theta \in \mathbb{R}$ is a location parameter and f_0 is a symmetric density with respect to the Lebesgue measure over the real line. Unless the resulting location model is exponential, existence of UMPU tests remains unclear in such cases. However, symmetry of the base density f_0 makes it most natural to restrict to level-α tests ϕ whose power function $\theta \mapsto \mathrm{E}_\theta[\phi]$ is symmetric about θ_0. As we will show, there are cases where *uniformly most powerful symmetric (UMPS)* tests—that is, tests that are uniformly most powerful in this class of level-α tests whose power function is symmetric about θ_0—can be constructed while the existence of UMPU tests remains an open question. The construction of UMPS tests relies on an original way to apply the *generalized Neyman–Pearson fundamental lemma*; see, e.g., Theorem 3.6.1 in Lehmann and Romano (2022). We also show that when UMPS tests do not exist, it may still be possible to construct UMP tests in the smaller class of level-α tests whose power function is symmetric and U-shaped. While we focus on single-observation, location, models for these results, we get rid of these restrictions when constructing UMPS tests in exponential models.

The outline of the paper is as follows. In Section 2, we consider single-observation location families and show that, under some structural condition on *symmetrized likelihood ratio (SLR)* functions, UMPS tests do exist. We provide examples that are not exponential, hence for which the existence of UMPU tests remains an open problem. In Section 3, we tackle an example where the aforementioned structural condition is not satisfied and prove that UMPS tests may then fail to exist. However, we provide another, weaker, structural condition on SLR functions under which *uniformly most powerful symmetric U-shaped (UMPSU)* tests do exist. In Section 4, we turn to multi-observation models, where we show that UMPS tests can be built in "symmetric exponential models", that is, in exponential models for which it is natural to impose symmetry of the power function. We tackle in particular an example of this type outside the framework of location families. In Section 5, we provide a wrap up and some final comments. Finally, an appendix collects proofs of some auxiliary results.

2 UMPS tests

Consider the model in which we observe a random variable X admitting a density of the form $f_\theta(x) = f_0(x-\theta)$ with respect to the Lebesgue measure on $\mathbb{R}$, where θ is a real number and f_0 is a fixed density satisfying $f_0(-x) = f_0(x)$ for any x. Thus, the location parameter $\theta \in \mathbb{R}$ is identified as the symmetry centre of the distribution of X. If this location model has monotone likelihood ratios, then UMP tests are available for the one-sided testing problems where one wants to test $\mathcal{H}_0 : \theta \leq \theta_0$ against $\mathcal{H}_1 : \theta > \theta_0$, or to test $\mathcal{H}_0 : \theta \geq \theta_0$ against $\mathcal{H}_1 : \theta < \theta_0$. For the corresponding two-sided problems, no UMP tests do exist and one classically aims at UMPU tests. If the model is not exponential, however, then no results guarantee the existence of UMPU tests. For instance, in the logistic case obtained with $f_0(x) = e^{-x}/(1+e^{-x})^2$, UMP tests can be constructed for one-sided problems since the monotone likelihood property is satisfied, but the existence of UMPU two-sided tests remains unclear.

Assume for simplicity that f_0 is non-vanishing, in the sense that $f_0(x) > 0$ for any x. In this framework, a key role will be played in the sequel by the *symmetrized likelihood ratio (SLR)* functions

$$x \mapsto h_\theta(x) := \frac{1}{2}\left(\frac{f_\theta(x)}{f_0(x)} + \frac{f_{-\theta}(x)}{f_0(x)}\right), \quad \theta > 0.$$

Note that symmetry of f_0 entails that, for any $\theta > 0$ and $x \in \mathbb{R}$,

$$\frac{f_\theta(-x)}{f_0(-x)} = \frac{f_{-\theta}(x)}{f_0(x)},$$

so that h_θ is symmetric about zero for any $\theta > 0$. We have the following result.

Theorem 1 *Consider the location model above and assume that f_0 is non-vanishing and is such that, for any $\theta > 0$, the SLR function h_θ is strictly increasing over $[0,\infty)$. Fix $\theta_0 \in \mathbb{R}$ and $\alpha \in (0,1)$. Then, (i) there exists $s_\alpha \geq 0$ such that the test defined by*

$$\phi_\alpha(x) = \begin{cases} 1 & \text{if } |x-\theta_0| > s_\alpha \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

satisfies $\mathrm{E}_{\theta_0}[\phi_\alpha] = \alpha$; (ii) the power function $\theta \mapsto \mathrm{E}_\theta[\phi_\alpha]$ is symmetric about θ_0; (iii) ϕ_α is UMPS at level α when testing $\mathcal{H}_0 : \theta = \theta_0$ against $\mathcal{H}_1 : \theta \neq \theta_0$.

As mentioned in the introduction, the proof is based on an original application of the generalized Neyman–Pearson fundamental lemma.

Proof of Theorem 1. Since (i)–(ii) result from trivial computations, we focus on the proof of (iii). Fix $\theta_1 \neq \theta_0$ arbitrarily. We will show that there exist $k_1, k_2 \geq 0$ such that the test ϕ_{k_1,k_2} defined by

$$\phi_{k_1,k_2}(x) = \begin{cases} 1 & \text{if } f_{\theta_1}(x) > k_1 f_{\theta_0}(x) + k_2(f_{\theta_1}(x) - f_{2\theta_0-\theta_1}(x)) \\ 0 & \text{otherwise} \end{cases}$$

satisfies $\mathrm{E}_{\theta_0}[\phi] = \alpha$ and $\mathrm{E}_{\theta_1}[\phi] - \mathrm{E}_{2\theta_0-\theta_1}[\phi] = 0$ (note that $2\theta_0 - \theta_1$ is the reflection of θ_1 with respect to θ_0). To do so, note that, for $k_2 = 1/2$, we have

$$\begin{aligned} g(x) &:= \frac{1}{f_{\theta_0}(x)}\big\{f_{\theta_1}(x) - k_1 f_{\theta_0}(x) - k_2(f_{\theta_1}(x) - f_{2\theta_0-\theta_1}(x))\big\} \\ &= (1-k_2)\frac{f_{\theta_1-\theta_0}(x-\theta_0)}{f_0(x-\theta_0)} + k_2\frac{f_{\theta_0-\theta_1}(x-\theta_0)}{f_0(x-\theta_0)} - k_1 \\ &= h_{|\theta_1-\theta_0|}(x-\theta_0) - k_1. \end{aligned}$$

The function $x \mapsto h_{|\theta_1-\theta_0|}(x-\theta_0)$ takes its values in $\mathbb{R}^+$ and is symmetric about θ_0. By assumption, it is strictly increasing on $[\theta_0,\infty)$. Thus, its restriction to $[\theta_0,\infty)$ is a one-to-one mapping from $[\theta_0,\infty)$ to $h_{|\theta_1-\theta_0|}([\theta_0,\infty))$. It follows that there exists $k_1 \geq 0$ such that $g(x) > 0$ if and only if $x \notin [\theta_0 - s_\alpha, \theta_0 + s_\alpha]$. For such a value of k_1 and $k_2 = 1/2$, the test ϕ_{k_1,k_2} thus coincides with ϕ_α in (1), hence satisfies $\mathrm{E}_{\theta_0}[\phi] = \alpha$ and $\mathrm{E}_{\theta_1}[\phi] - \mathrm{E}_{2\theta_0-\theta_1}[\phi] = 0$. It then follows from Theorem 3.6.1(iii) in Lehmann and Romano (2022) that ϕ_α is most powerful when testing $\mathcal{H}_0 : \theta = \theta_0$ against $\mathcal{H}_1 : \theta = \theta_1$ in the class of level-α tests such that $\mathrm{E}_{2\theta_0-\theta_1}[\phi] \leq \mathrm{E}_{\theta_1}[\phi]$, hence also most powerful for the same problem in the class of level-α tests such that $\mathrm{E}_{2\theta_0-\theta_1}[\phi] = \mathrm{E}_{\theta_1}[\phi]$.

Now, let $\mathcal{C}^{\mathrm{s}}_\alpha$ be the class of tests ϕ for $\mathcal{H}_0 : \theta = \theta_0$ against $\mathcal{H}_1 : \theta \neq \theta_0$ that have level α and have a power function that is symmetric about θ_0. Obviously, Parts (i)–(ii) of the result ensure that $\phi_\alpha \in \mathcal{C}^{\mathrm{s}}_\alpha$. Fix then $\phi \in \mathcal{C}^{\mathrm{s}}_\alpha$ and an arbitrary $\theta_1 \neq \theta_0$. Since ϕ has level α when testing $\mathcal{H}_0 : \theta = \theta_0$ against $\mathcal{H}_1 : \theta = \theta_1$ and satisfies $\mathrm{E}_{2\theta_0-\theta_1}[\phi] = \mathrm{E}_{\theta_1}[\phi]$, we must have $\mathrm{E}_{\theta_1}[\phi_\alpha] \geq \mathrm{E}_{\theta_1}[\phi]$. Since $\theta_1 \neq \theta_0$ was arbitrary, we conclude that $\mathrm{E}_{\theta_1}[\phi_\alpha] \geq \mathrm{E}_{\theta_1}[\phi]$ for any $\theta_1 \neq \theta_0$, which establishes the result. □

As an example, consider the logistic case above. Direct computations show that, for any $\theta > 0$, the resulting SLR function

$$x \mapsto h_\theta(x) = \frac{e^{-\theta}}{2}\left\{\left(\frac{e^x+1}{e^x+e^{-\theta}}\right)^2 + \left(\frac{e^{-x}+1}{e^{-x}+e^{-\theta}}\right)^2\right\}$$

has a positive derivative over $(0,\infty)$, hence is strictly increasing over $[0,\infty)$; see Figure 1. It thus follows from Theorem 1 that the test defined by

$$\phi_\alpha(x) = \begin{cases} 1 & \text{if } |x-\theta_0| > \ln(\frac{2-\alpha}{\alpha}) \\ 0 & \text{otherwise} \end{cases}$$

is UMPS at level α for $\mathcal{H}_0 : \theta = \theta_0$ against $\mathcal{H}_1 : \theta \neq \theta_0$. A plot of the corresponding power function, namely

$$\theta \mapsto \mathrm{E}_\theta[\phi_\alpha] = \sum_{s\in\{-1,1\}} \frac{\alpha}{\alpha + (2-\alpha)\exp(s|\theta-\theta_0|)},$$

is provided for $\theta_0 = 0$ and $\alpha = 5\%$ in the left panel of Figure 2. We stress that, since this model is not exponential, it is unknown whether or not this test is UMPU at level α for the same testing problem.

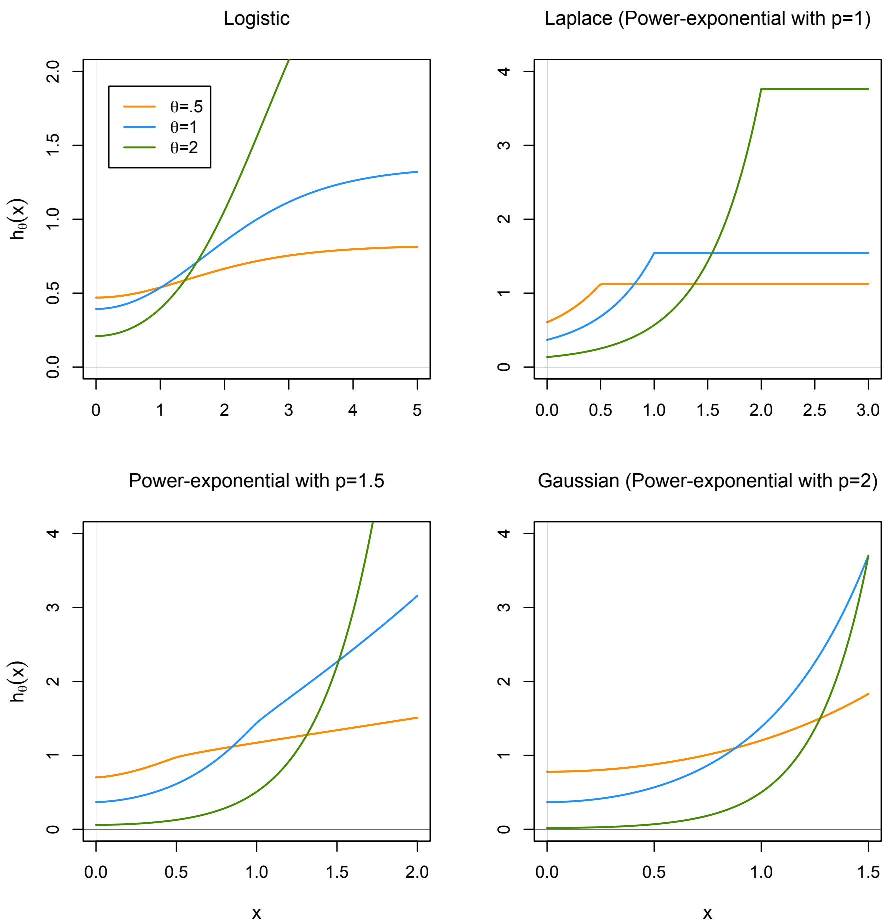

Fig. 1 Plots of the SLR functions $x \mapsto h_\theta(x)$ over positive values of x for $\theta = .5$, $\theta = 1$ and $\theta = 2$, when f_0 is the logistic density and the power-exponential densities with $p = 1$ (Laplace density), $p = 1.5$, and $p = 2$ (Gaussian density).

Other examples that are compatible with the result above are given by the power-exponential densities $f_0(x) = c_p \exp(-|x|^p)$, with $p \in (1, 2]$, where $c_p = p/(2\Gamma(\frac{1}{p}))$ is a normalizing constant. For any $\theta > 0$, the resulting SLR function

$$x \mapsto h_\theta(x) = \frac{1}{2}(e^{|x|^p - |x-\theta|^p} + e^{|x|^p - |x+\theta|^p}) \tag{2}$$

is strictly increasing over $[0, \infty)$; this is illustrated in Figure 1 and proved in Section 6 (see Proposition 5). Therefore, Theorem 1 ensures that the two-sided test ϕ_α is again UMPS at level α. Note that the case of the Laplace

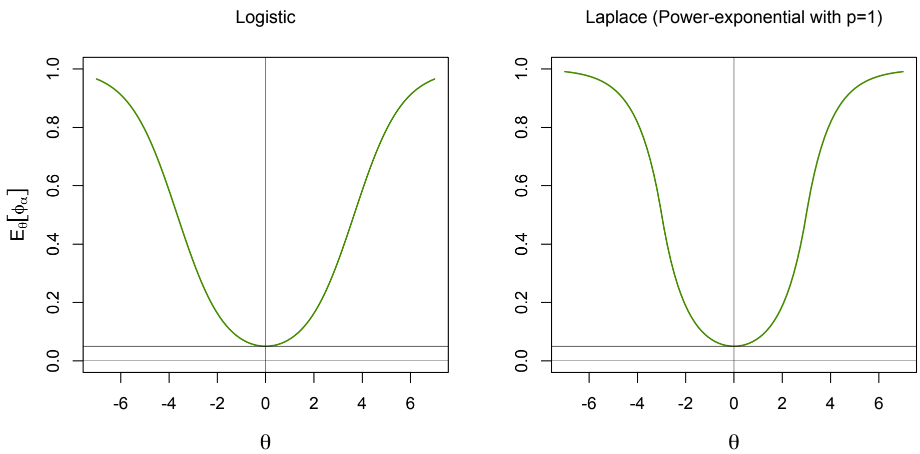

Fig. 2 Plots of the power function $\theta \mapsto \mathrm{E}_\theta[\phi_\alpha]$ in the logistic case (left) and Laplace case (right), both for $\theta_0 = 0$ and $\alpha = 5\%$.

distribution, that is obtained for $p = 1$, is of a different nature: since the SLR function in (2) is not strictly increasing over $[0, \infty)$ for $p = 1$ (it is only non-decreasing; see Figure 1), Theorem 1 does not apply. Interestingly, the two-sided test ϕ_α is still UMPS at level α, as the following result shows.

Proposition 1 *Consider the location model associated with the Laplace density $f_0(x) = \frac{1}{2}\exp(-|x|)$. Fix $\theta_0 \in \mathbb{R}$ and $\alpha \in (0,1)$. Then, (i) the test defined by*

$$\phi_\alpha(x) = \begin{cases} 1 & \text{if } |x - \theta_0| > \ln(\frac{1}{\alpha}) \\ 0 & \text{otherwise} \end{cases}$$

satisfies $\mathrm{E}_{\theta_0}[\phi_\alpha] = \alpha$ and (ii) is UMPS at level α when testing $\mathcal{H}_0 : \theta = \theta_0$ against $\mathcal{H}_1 : \theta \neq \theta_0$.

Proof of Proposition 1. Since Part (i) of the result is trivial, we focus on the proof of Part (ii). Fix $\theta_1 \neq \theta_0$ arbitrarily, and note that, for $k_2 = 1/2$,

$$\begin{aligned} g(x) &:= \frac{1}{f_{\theta_0}(x)}\big\{f_{\theta_1}(x) - k_1 f_{\theta_0}(x) - k_2(f_{\theta_1}(x) - f_{2\theta_0-\theta_1}(x))\big\} \\ &= (1-k_2)\frac{f_{\theta_1-\theta_0}(x-\theta_0)}{f_0(x-\theta_0)} + k_2\frac{f_{\theta_0-\theta_1}(x-\theta_0)}{f_0(x-\theta_0)} - k_1 \\ &= \begin{cases} \frac{1}{2}(e^{2|x-\theta_0|-|\theta_1-\theta_0|} + e^{-|\theta_1-\theta_0|}) - k_1 & \text{if } |x-\theta_0| \le |\theta_1-\theta_0| \\ \cosh(|\theta_1-\theta_0|) - k_1 & \text{otherwise,} \end{cases} \end{aligned}$$

where cosh is the hyperbolic cosine function. We consider two cases:

(a) $|\theta_1 - \theta_0| > \ln(\frac{1}{\alpha})$. Then, with $k_1 = (e^{2\ln(1/\alpha)-|\theta_1-\theta_0|} + e^{-|\theta_1-\theta_0|})/2$, the test ϕ_α in the statement of the theorem is of the form

$$\phi_{k_1,k_2}(x) = \begin{cases} 1 & \text{if } f_{\theta_1}(x) > k_1 f_{\theta_0}(x) + k_2(f_{\theta_1}(x) - f_{2\theta_0-\theta_1}(x)) \\ 0 & \text{otherwise} \end{cases}$$

and satisfies $\mathrm{E}_{\theta_0}[\phi] = \alpha$ and $\mathrm{E}_{\theta_1}[\phi] - \mathrm{E}_{2\theta_0-\theta_1}[\phi] = 0$. Since $k_1, k_2 \geq 0$, Theorem 3.6.1(iii) in Lehmann and Romano (2022) entails that ϕ_α is most powerful when testing $\mathcal{H}_0 : \theta = \theta_0$ against $\mathcal{H}_1 : \theta = \theta_1$ in the class of level-α tests such that $\mathrm{E}_{\theta_1}[\phi] - \mathrm{E}_{2\theta_0-\theta_1}[\phi] \leq 0$.

(b) $|\theta_1 - \theta_0| \leq \ln(\frac{1}{\alpha})$. Then, with $k_1 = \cosh(\theta_1 - \theta_0)(> 0)$, the test

$$\begin{aligned} \frac{1}{2}&(e^{2|x-\theta_0|-|\theta_1-\theta_0|} + e^{-|\theta_1-\theta_0|}) - (e^{|\theta_1-\theta_0|} + e^{-|\theta_1-\theta_0|})/2 \\ &= \frac{1}{2}(e^{2|x-\theta_0|-|\theta_1-\theta_0|}) - e^{|\theta_1-\theta_0|}/2 \\ &= (e^{2|x-\theta_0|-2|\theta_1-\theta_0|}) - 1)e^{|\theta_1-\theta_0|}/2 \end{aligned}$$

$$\phi_{k_1,k_2}(x) = \begin{cases} 1 & \text{if } g(x) > 0 \\ \alpha e^{|\theta_1-\theta_0|} & \text{if } g(x) = 0 \\ 0 & \text{if } g(x) < 0 \end{cases}$$

rewrites

$$\phi_{\theta_1}(x) = \begin{cases} \alpha e^{|\theta_1-\theta_0|} & \text{if } |x-\theta_0| \geq |\theta_1 - \theta_0| \\ 0 & \text{otherwise} \end{cases}$$

and satisfies both $\mathrm{E}_{\theta_0}[\phi] = \alpha$ and $\mathrm{E}_{\theta_1}[\phi] - \mathrm{E}_{2\theta_0-\theta_1}[\phi] = 0$. As above, Theorem 3.6.1(iii) in Lehmann and Romano (2022) then entails that ϕ_{θ_1} is most powerful when testing $\mathcal{H}_0 : \theta = \theta_0$ against $\mathcal{H}_1 : \theta = \theta_1$ in the class of level-α tests such that $\mathrm{E}_{\theta_1}[\phi] - \mathrm{E}_{2\theta_0-\theta_1}[\phi] \leq 0$. A direct computation, however, shows that

$$\mathrm{E}_{\theta_1}[\phi_\alpha] = \alpha \cosh(\theta_1 - \theta_0) = \mathrm{E}_{\theta_1}[\phi_{\theta_1}],$$

so that ϕ_α itself is most powerful when testing $\mathcal{H}_0 : \theta = \theta_0$ against $\mathcal{H}_1 : \theta = \theta_1$ in the class of level-α tests such that $\mathrm{E}_{\theta_1}[\phi] - \mathrm{E}_{2\theta_0-\theta_1}[\phi] \leq 0$.

Thus, we showed that, irrespective of $\theta_1 \neq \theta_0$, the test ϕ_α, which does not depend on θ_1, is most powerful when testing $\mathcal{H}_0 : \theta = \theta_0$ against $\mathcal{H}_1 : \theta = \theta_1$ in the class of level-α tests such that $\mathrm{E}_{\theta_1}[\phi] - \mathrm{E}_{2\theta_0-\theta_1}[\phi] \leq 0$, hence also in the smaller class of level-α tests such that $\mathrm{E}_{\theta_1}[\phi] - \mathrm{E}_{2\theta_0-\theta_1}[\phi] = 0$. The argument used to conclude the proof of Theorem 1 thus establishes that ϕ_α is uniformly most powerful for $\mathcal{H}_0 : \theta = \theta_0$ against $\mathcal{H}_1 : \theta \neq \theta_0$ in the class of level-α tests whose power function is symmetric about θ_0. □

It appears difficult to generalize Theorem 1 to show that the natural two-sided test remains UMPS when SLR functions are only monotone non-decreasing (rather than monotone strictly increasing) over $[0, \infty)$. Interestingly, monotonicity cannot be dropped, though, as we will see in Section 3.1 below.

3 UMPSU tests

It might be tempting to conjecture that the natural two-sided test is always UMPS. In Section 3.1, we treat an example, involving SLR functions that are not monotone non-decreasing over $[0, \infty)$, in which the natural two-sided test actually fails to be UMPS for some significance levels. In Section 3.2, we then prove that, under a weaker condition than the one ensuring existence of UMPS tests in Theorem 1, there exist *uniformly most powerful symmetric U-shaped (UMPSU)* tests, that is, tests that are UMP in the class of level-α tests whose power function is symmetric and U-shaped.

3.1 A negative example regarding UMPS testing

We consider the Cauchy model obtained with $f_0(x) = 1/(\pi(1+x^2))$, for which it is easy to check that, for any $\theta > 0$, there exists $r_\theta > 0$ such that the corresponding SLR function h_θ is monotone strictly increasing on $[0, r_\theta]$, then monotone strictly decreasing on $[r_\theta, \infty)$; see Figure 1. As the following result shows, this violation of the monotonicity condition affects the UMPS nature of the two-sided test at all usual significance levels α, whereas, for some other significance levels, this test remains UMPS.

Proposition 2 *Consider the location model associated with the Cauchy density $f_0(x) = 1/(\pi(1+x^2))$. Fix $\theta_0 \in \mathbb{R}$ and $\alpha \in (0, 1)$. Then, (i) the test defined by*

$$\phi_\alpha(x) = \begin{cases} 1 & \textit{if } |x - \theta_0| > \tan\left(\frac{\pi(1-\alpha)}{2}\right) \\ 0 & \textit{otherwise} \end{cases}$$

satisfies $\mathrm{E}_{\theta_0}[\phi_\alpha] = \alpha$; (ii) for $\alpha \in [\frac{2}{3}, 1)$, this test is UMPS at level α when testing $\mathcal{H}_0 : \theta = \theta_0$ against $\mathcal{H}_1 : \theta \neq \theta_0$. (iii) For $\alpha \in (0, \frac{2}{3})$, letting

$$t_\alpha := \sqrt{3\tan^2\left(\tfrac{\pi(1-\alpha)}{2}\right) - 1}, \tag{3}$$

the test ϕ_α is UMPS at level α when testing $\mathcal{H}_0 : \theta = \theta_0$ against $\mathcal{H}_1 : \theta \notin (\theta_0 - t_\alpha, \theta_0 + t_\alpha)$; however, for any $\theta_1 \in (\theta_0 - t_\alpha, \theta_0 + t_\alpha) \setminus \{\theta_0\}$, there exists a test ϕ_{θ_1} in the class of level-α tests whose power function is symmetric about θ_0 that provides $\mathrm{E}_{\theta_1}[\phi_{\theta_1}] > \mathrm{E}_{\theta_1}[\phi_\alpha]$.

Proof of Proposition 2. Since Part (i) of the result follows from straightforward computations, we will only prove Parts (ii)–(iii). Before doing so, note that, with $k_2 = 1/2$, we have, for any $\theta_1 \neq \theta_0$,

$$
\begin{aligned}
g(x) &:= \frac{1}{f_{\theta_0}(x)} \big\{ f_{\theta_1}(x) - k_1 f_{\theta_0}(x) - k_2 (f_{\theta_1}(x) - f_{2\theta_0-\theta_1}(x)) \big\} \\
&= (1-k_2) \frac{f_{\theta_1-\theta_0}(x-\theta_0)}{f_0(x-\theta_0)} + k_2 \frac{f_{\theta_0-\theta_1}(x-\theta_0)}{f_0(x-\theta_0)} - k_1 \\
&= \frac{1}{2} \left(\frac{1+(x-\theta_0)^2}{1+(x-\theta_1)^2} + \frac{1+(x-\theta_0)^2}{1+(x-2\theta_0+\theta_1)^2} \right) - k_1 \\
&=: \ell_{\theta_1}(x) - k_1,
\end{aligned}
$$

say (we do not stress dependence of ℓ_{θ_1} on θ_0, that is fixed throughout). Since the function ℓ_{θ_1} will play a key role in this proof, we list some of its properties here: first, it is continuous and symmetric about θ_0. Second, there exists x_0 (depending on θ_0 and θ_1) such that ℓ_{θ_1} is strictly increasing on $[\theta_0, x_0]$ and strictly decreasing on $[x_0, \infty)$. We have $\ell_{\theta_1}(\theta_0) = 1/(1+(\theta_1-\theta_0)^2) \in (0,1)$ and, since $\ell_{\theta_1}(x) \to 1$ as $x \to \infty$, we must have $\ell_{\theta_1}(x_0) > 1$. Since the only solutions of $\ell_{\theta_1}(x) = 1$ are

$$
\theta_0 \pm c_{\theta_1}, \qquad \text{with } c_{\theta_1} := \sqrt{\frac{1+(\theta_1-\theta_0)^2}{3}},
$$

we then have that, for any $t \in [0, c_{\theta_1}]$, there exists $k_1 > 0$ such that $g(x) = \ell_{\theta_1}(x) - k_1 > 0$ (resp., $=$ or $<$), if and only if $x \notin [\theta_0 - t, \theta_0 + t]$ (resp., $x \in \{\theta_0 - t, \theta_0 + t\}$ or $x \in (\theta_0 - t, \theta_0 + t)$). We can now proceed with the proof of (ii)–(iii).

(ii) Fix $\alpha \in [\frac{2}{3}, 1)$ and an arbitrary $\theta_1 \neq \theta_0$. Since we then have

$$
\tan\Big(\frac{\pi(1-\alpha)}{2}\Big) \leq \tan\Big(\frac{\pi}{6}\Big) = \frac{1}{\sqrt{3}} \leq c_{\theta_1},
$$

there exist $k_1, k_2 > 0$ (actually, $k_2 = 1/2$) such that the test ϕ_α in the statement of the result coincides with the test defined by

$$
\phi_{k_1,k_2}(x) = \begin{cases} 1 & \text{if } g(x) > 0 \\ 0 & \text{otherwise,} \end{cases} \tag{4}
$$

which, as in the previous proofs, implies that ϕ_α is most powerful when testing $\mathcal{H}_0 : \theta = \theta_0$ against $\mathcal{H}_1 : \theta = \theta_1$ in the class of level-α tests such that $\mathrm{E}_{\theta_1}[\phi] = \mathrm{E}_{2\theta_0-\theta_1}[\phi]$. Since this test does not depend on θ_1, it is then also most powerful when testing $\mathcal{H}_0 : \theta = \theta_0$ against $\mathcal{H}_1 : \theta \neq \theta_0$ in the class of level-α tests such that $\mathrm{E}_{\theta_1}[\phi] = \mathrm{E}_{2\theta_0-\theta_1}[\phi]$. Part (ii) of the result thus follows in the same way as in the proof of Theorem 1.

(iii) Fix $\alpha \in (0, \frac{2}{3})$ and an arbitrary $\theta_1 \notin (\theta_0 - t_\alpha, \theta_0 + t_\alpha)$. Then,

$$
\tan\Big(\frac{\pi(1-\alpha)}{2}\Big) = \sqrt{\frac{1+t_\alpha^2}{3}} \leq \sqrt{\frac{1+(\theta_1-\theta_0)^2}{3}} = c_{\theta_1},
$$

so that, as above, there exist $k_1, k_2 > 0$ (again, $k_2 = 1/2$) such that the test ϕ_α in the statement of the result coincides with the test in (4), hence is most powerful when testing $\mathcal{H}_0 : \theta = \theta_0$ against $\mathcal{H}_1 : \theta = \theta_1$ in the class of level-α tests such that $\mathrm{E}_{\theta_1}[\phi] = \mathrm{E}_{2\theta_0-\theta_1}[\phi]$, which, in turn, implies that ϕ_α is uniformly most powerful for $\mathcal{H}_0 : \theta = \theta_0$ against $\mathcal{H}_1 : \theta \notin (\theta_0 - t_\alpha, \theta_0 + t_\alpha)$ in the class of level-α tests whose power function is symmetric about θ_0.

Still with $\alpha \in (0, \frac{2}{3})$, fix then an arbitrary $\theta_1 \in (\theta_0 - t_\alpha, \theta_0 + t_\alpha) \setminus \{\theta_0\}$. Here, still with $k_2 = 1/2$ throughout, any value of $k_1 \in [\ell_{\theta_1}(\theta_0), 1]$ will provide a test ϕ_{k_1,k_2} in (4) that yields

$$\mathrm{E}_{\theta_0}[\phi_{k_1,k_2}] \geq \mathrm{P}_{\theta_0}[|X - \theta_0| > c_{\theta_1}] > \mathrm{P}_{\theta_0}\big[|X - \theta_0| > \tan\big(\tfrac{\pi(1-\alpha)}{2}\big)\big] = \alpha,$$

so that ϕ_{k_1,k_2} does *not* satisfy the constraint $\mathrm{E}_{\theta_0}[\phi] = \alpha$. Obviously, with $k_1 = \ell_{\theta_1}(x_0)$, we have $\mathrm{E}_{\theta_0}[\phi_{k_1,k_2}] = 0$. Since $\mathrm{E}_{\theta_0}[\phi_{k_1,k_2}]$ depends continuously on k_1, there exists $k_1 \in (1, \ell_{\theta_1}(x_0))$ such that $\mathrm{E}_{\theta_0}[\phi_{k_1,k_2}] = \alpha$. The properties of ℓ_{θ_1} actually imply that k_1 is unique, and that the resulting test ϕ_{k_1,k_2} rewrites

$$\phi_{k_1,k_2}(x) = \begin{cases} 1 & \text{if } |x - \theta_0| \in (a^-_{\theta_1}, a^+_{\theta_1}) \\ 0 & \text{otherwise,} \end{cases} \tag{5}$$

where $a^\pm_{\theta_1} > 0$ are such that

$$\begin{aligned}(a^\pm_{\theta_1})^2 = {} & \frac{(k_1 + \frac{1}{2})(\theta_1 - \theta_0)^2 - k_1 + 1}{k_1 - 1} \\ & \pm \frac{|\theta_1 - \theta_0|\sqrt{(2k_1 + \frac{1}{4})(\theta_1 - \theta_0)^2 - 4k_1(k_1 - 1)}}{k_1 - 1}\end{aligned} \tag{6}$$

involve the (unique) value k_1 for which

$$\begin{aligned}\mathrm{E}_{\theta_0}[\phi_{k_1,k_2}] &= 2\mathrm{P}_{\theta_0}[a^-_{\theta_1} < |X - \theta_0| < a^+_{\theta_1}] \\ &= \frac{2}{\pi}\big(\arctan(a^+_{\theta_1}) - \arctan(a^-_{\theta_1})\big) \\ &= \alpha.\end{aligned} \tag{7}$$

It follows from Theorem 3.6.1(iii) in Lehmann and Romano (2022) that the test $\phi_{\theta_1} := \phi_{k_1,k_2}$ in (5) is most powerful when testing $\mathcal{H}_0 : \theta = \theta_0$ against $\mathcal{H}_1 : \theta = \theta_1$ in the class of level-α tests such that $\mathrm{E}_{\theta_1}[\phi] = \mathrm{E}_{2\theta_0-\theta_1}[\phi]$. Since ϕ_α belongs to this class, we thus have $\mathrm{E}_{\theta_1}[\phi_{\theta_1}] \geq \mathrm{E}_{\theta_1}[\phi_\alpha]$. Now, assume that $\mathrm{E}_{\theta_1}[\phi_{\theta_1}] = \mathrm{E}_{\theta_1}[\phi_\alpha]$. According to Theorem 3.6.1(iv) from Lehmann and Romano (2022), we must then have that

$$\phi_\alpha(x) = \begin{cases} 1 & \text{if } |x - \theta_0| \in (a^-_{\theta_1}, a^+_{\theta_1}) \\ 0 & \text{otherwise} \end{cases}$$

almost everywhere with respect to the Lebesgue measure. This is, however, not the case since $\phi_\alpha(x) = 1 \neq 0$ for any x in the set $(\max(a^+_{\theta_1}, \tan(\pi(1-\alpha)/2)), \infty)$. Thus, we have $\mathrm{E}_{\theta_1}[\phi_{\theta_1}] > \mathrm{E}_{\theta_1}[\phi_\alpha]$, as was to be proved. $\square$

The proof of Proposition 2(iii) identifies, for any $\theta_1 \in (\theta_0 - t_\alpha, \theta_0 + t_\alpha)\backslash\{\theta_0\}$, a test ϕ_{θ_1} that is most powerful for $\mathcal{H}_0 : \theta = \theta_0$ against $\mathcal{H}_1 : \theta = \theta_1$ in the class of level-α tests whose power function is symmetric about θ_0 (note that the power function of each such ϕ_{θ_1} is indeed symmetric about θ_0). The power function of ϕ_{θ_1} is

$$\theta \mapsto \mathrm{E}_\theta[\phi_{\theta_1}] = \frac{1}{\pi} \sum_{s \in \{-1,1\}} \big(\arctan(a^+_{\theta_1} + s|\theta - \theta_0|) - \arctan(a^-_{\theta_1} + s|\theta - \theta_0|) \big),$$

where the positive real numbers $a^\pm_{\theta_1}$ are defined through (6)–(7). This generates the *symmetric power envelope function*

$$\theta \mapsto \mathrm{PE}(\theta) := \begin{cases} \mathrm{E}_\theta[\phi_\theta] & \text{if } |\theta - \theta_0| \in (0, t_\alpha) \\ \mathrm{E}_\theta[\phi_\alpha] & \text{otherwise}, \end{cases} \tag{8}$$

which is to be compared with the power function of ϕ_α, namely

$$\begin{aligned} \theta \mapsto \mathrm{E}_\theta[\phi_\alpha] &= \mathrm{P}_\theta\big[|X - \theta_0| > \tan\big(\tfrac{\pi(1-\alpha)}{2}\big)\big] \\ &= 1 - \frac{1}{\pi} \sum_{s \in \{-1,1\}} \arctan\big(\tan\big(\tfrac{\pi(1-\alpha)}{2}\big) + s|\theta - \theta_0|\big). \end{aligned}$$

Figure 3 plots, in the Cauchy case, for $\theta_0 = 0$ and $\alpha = 5\%$, the power function of the test ϕ_α, the symmetric power envelope function in (8), and the power function of four of the tests ϕ_{θ_1} that are generating this envelope function. As described by Proposition 2, the power function of ϕ_α achieves the symmetric power envelope outside $(-t_\alpha, t_\alpha)$, but the figure reveals that the power deficit of ϕ_α with respect to this envelope is quite severe in $(-t_\alpha, t_\alpha)$.

3.2 UMPSU tests

In the Cauchy example considered above, the tests ϕ_{θ_1} for $0 < |\theta_1 - \theta_0| < t_\alpha$ prevent the existence of a UMPS test at level α. While they have a symmetric power function, these tests ϕ_{θ_1} are not reasonable decision rules since their power function is not monotone non-decreasing in $|\theta_1 - \theta_0|$. This suggests restricting to the class of level-α tests whose power function is symmetric in θ and monotone non-decreasing in $|\theta - \theta_0|$, or, equivalently, to the class of level-α tests whose power function is symmetric and U-shaped in θ. In the sequel, a test that is uniformly most powerful in this class will be said to be a *uniformly most powerful symmetric U-shaped (UMPSU)* test.

Of course, the power function of the natural two-sided test ϕ_α is always symmetric about θ_0. The next result shows that, at least when f_0 is unimodal, this power function is also U-shaped.

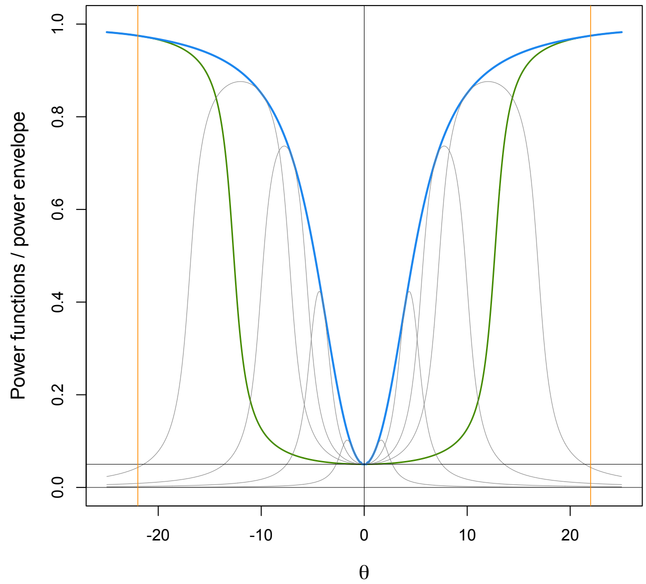

Fig. 3 Plots, in the Cauchy case, for $\theta_0 = 0$ and $\alpha = 5\%$, of the power function $\theta \mapsto \mathrm{E}_\theta[\phi_\alpha]$ of the UMPSU test (green), of the power envelope function in (8) (blue), and of the power functions $\theta \mapsto \mathrm{E}_\theta[\phi_{\theta_1}]$ associated with $\theta_1 = 1, 4, 7, 10$ (grey). The vertical orange lines indicate the values of $\pm t_\alpha$ in (3) and the horizontal grey line corresponds to the nominal level α.

Proposition 3 *Consider the location model with a base symmetric density f_0 that is Riemann-integrable and fix $\theta_0 \in \mathbb{R}$. (i) If f_0 is unimodal, then, for any $\alpha \in (0,1)$, the test ϕ_α defined in Theorem 1 has a power function that is symmetric about θ_0 and U-shaped. (ii) Assuming further that f_0 is continuous, we have that if f_0 is not unimodal, then there exists $\alpha \in (0,1)$ such that the power function of ϕ_α fails to be U-shaped.*

It follows from this result that, when f_0 is unimodal (as, e.g., in the Cauchy case), ϕ_α is a possible candidate to be a UMPSU test at level α. We now provide a sufficient condition under which ϕ_α indeed enjoys this optimality property.

Theorem 2 *Fix $\theta_0 \in \mathbb{R}$ and $\alpha \in (0,1)$. Consider the location model and assume that the base symmetric density f_0 is non-vanishing, Riemann-integrable, and satisfies the following assumption: there exists $M > 0$ such that, for any $\theta \geq M$,*

$$\{x \in \mathbb{R}^+ : h_\theta(x) \leq h_\theta(s_\alpha)\} = [0, s_\alpha], \tag{9}$$

where s_α is as in Theorem 1. Then, the test ϕ_α defined in that result is UMPSU at level α when testing $\mathcal{H}_0 : \theta = \theta_0$ against $\mathcal{H}_1 : \theta \neq \theta_0$.

Proof of Theorem 2. It follows from Theorem 1(i) and Proposition 3(i) that ϕ_α is a level-α test whose power function is symmetric about θ_0 and U-shaped. Fix then $\theta_1 \neq \theta_0$ arbitrarily. Take $c > 1$ large enough to let $\theta_{1,c} := \theta_0 + c(\theta_1 - \theta_0)$ be such that $|\theta_{1,c} - \theta_0| \geq M$. Define then $r_{\theta_{1,c}} := \mathrm{E}_{\theta_{1,c}}[\phi_\alpha] - \mathrm{E}_{\theta_1}[\phi_\alpha]$; since the power function of ϕ_α is symmetric about θ_0 and U-shaped, we have that $r_{\theta_{1,c}} \geq 0$. We will show that there exist $k_1, k_2, k_3, k_4 \geq 0$ such that the test $\phi_k = \phi_{k_1,k_2,k_3,k_4}$ defined by

$$\phi_k(x) = \begin{cases} 1 & \text{if } g(x) > 0 \\ 0 & \text{otherwise,} \end{cases} \tag{10}$$

with

$$\begin{aligned} g(x) := \frac{1}{f_{\theta_0}(x)} \big\{ & f_{\theta_1}(x) - k_1 f_{\theta_0}(x) - k_2(f_{\theta_1}(x) - f_{2\theta_0-\theta_1}(x)) \\ & - k_3(f_{\theta_{1,c}}(x) - f_{2\theta_0-\theta_{1,c}}(x)) - k_4(f_{\theta_1}(x) - f_{\theta_{1,c}}(x)) \big\}, \end{aligned}$$

satisfies

$$\mathrm{E}_{\theta_0}[\phi_k] = \int \phi_k(x) f_{\theta_0}(x)\, dx = \alpha, \tag{11}$$

$$\mathrm{E}_{\theta_1}[\phi_k] - \mathrm{E}_{2\theta_0-\theta_1}[\phi_k] = \int \phi_k(x)(f_{\theta_1}(x) - f_{2\theta_0-\theta_1}(x))\, dx = 0, \tag{12}$$

$$\mathrm{E}_{\theta_{1,c}}[\phi_k] - \mathrm{E}_{2\theta_0-\theta_{1,c}}[\phi_k] = \int \phi_k(x)(f_{\theta_{1,c}}(x) - f_{2\theta_0-\theta_{1,c}}(x))\, dx = 0 \tag{13}$$

and

$$\mathrm{E}_{\theta_1}[\phi_k] - \mathrm{E}_{\theta_{1,c}}[\phi_k] = \int \phi_k(x)(f_{\theta_1}(x) - f_{\theta_{1,c}}(x))\, dx = r_{\theta_{1,c}}. \tag{14}$$

Letting $k_1 = h_{|\theta_{1,c}-\theta_0|}(s_\alpha)$, $k_2 = 0$, $k_3 = \frac{1}{2}$ and $k_4 = 1$, we have

$$\begin{aligned} g(x) &= \frac{1}{2}\Big(\frac{f_{\theta_{1,c}}(x)}{f_{\theta_0}(x)} + \frac{f_{2\theta_0-\theta_{1,c}}(x)}{f_{\theta_0}(x)} \Big) - h_{|\theta_{1,c}-\theta_0|}(s_\alpha) \\ &= \frac{1}{2}\Big(\frac{f_{\theta_{1,c}-\theta_0}(x-\theta_0)}{f_0(x-\theta_0)} + \frac{f_{\theta_0-\theta_{1,c}}(x-\theta_0)}{f_0(x-\theta_0)} \Big) - h_{|\theta_{1,c}-\theta_0|}(s_\alpha) \\ &= h_{|\theta_{1,c}-\theta_0|}(x-\theta_0) - h_{|\theta_{1,c}-\theta_0|}(s_\alpha), \end{aligned}$$

so that, in view of the assumption in (9), $\phi_k = \phi_\alpha$. It follows that ϕ_k satisfies (11)–(14) (since this is indeed the case for ϕ_α) and, from Theorem 3.6.1(iii) in Lehmann and Romano (2022), that ϕ_α is UMP when testing θ_0 against θ_1 in the class of tests satisfying

$$\mathrm{E}_{\theta_0}[\phi] = \int \phi(x) f_{\theta_0}(x)\, dx \leq \alpha,$$

$$\mathrm{E}_{\theta_1}[\phi] - \mathrm{E}_{2\theta_0-\theta_1}[\phi] = \int \phi(x)(f_{\theta_1}(x) - f_{2\theta_0-\theta_1}(x))\, dx = 0,$$

$$\mathrm{E}_{\theta_{1,c}}[\phi] - \mathrm{E}_{2\theta_0-\theta_{1,c}}[\phi] = \int \phi(x)(f_{\theta_{1,c}}(x) - f_{2\theta_0-\theta_{1,c}}(x))\, dx = 0$$

and

$$\mathrm{E}_{\theta_1}[\phi] - \mathrm{E}_{\theta_{1,c}}[\phi] = \int \phi(x)(f_{\theta_1}(x) - f_{\theta_{1,c}}(x))\, dx \leq r_{\theta_1,c},$$

hence also in the (smaller) class of level-α tests whose power function is symmetric about θ_0 and U-shaped. Since ϕ_α belongs to this class and does not depend on the arbitrary value θ_1, the result follows. □

Figure 4 provides the plots of SLR functions associated with various densities that do not satisfy the monotonicity condition from Theorem 1 yet satisfy the much weaker condition (9) from Theorem 2. In each case, it is seen that, for θ large enough, the SLR functions h_θ are strictly increasing in an interval $[0, c_\theta]$ with $h_\theta(c_\theta) = 1$ and that $h_\theta(x)$ then is strictly larger than one for any $x > c_\theta$. Since c_θ diverges to infinity as θ does, the condition (9) is satisfied with $M = \inf\{\theta > 0 : c_\theta \geq s_\alpha\}$. Of course, it should be shown formally that SLR functions indeed behave in this way, but this was already shown for the Cauchy case in the proof of Proposition 2. As a corollary, the natural two-sided test ϕ_α from this proposition is UMPSU at level α when testing $\mathcal{H}_0 : \theta = \theta_0$ against $\mathcal{H}_1 : \theta \neq \theta_0$.

4 UMPS tests in exponential models

Turning to the multi-observation case, we now assume that the model is exponential, in the sense that there exists a σ-finite dominating measure ν such that the corresponding densities take the form

$$f_\theta(x) = \frac{d\mathrm{P}_\theta}{d\nu}(x) = C(\theta)h(x)\exp(\eta(\theta)T(x)). \tag{15}$$

Constructing UMPS tests does only make sense when the underlying model enjoys some symmetry itself. In the present context, we assume that the exponential family at hand is symmetric about θ_0 in the following sense.

Definition 1 Let the parameter value θ_0 be such that $2\theta_0 - \theta \in \Theta$ for any $\theta \in \Theta$. Then, we will say that the model described by the densities in (15) is *symmetric about* θ_0 if and only if the distribution of $T - \mathrm{E}_{\theta_0}[T]$ under $\mathrm{P}_{2\theta_0-\theta}$ is the same as the distribution of $-(T - \mathrm{E}_{\theta_0}[T])$ under P_θ.

Note that this symmetry property implies in particular that the distribution of T under P_{θ_0} is symmetric with respect to its mean $\mathrm{E}_{\theta_0}[T]$. The following result provides a necessary and sufficient condition for symmetry of exponential models, that will play a role in the proof of Theorem 3 below.

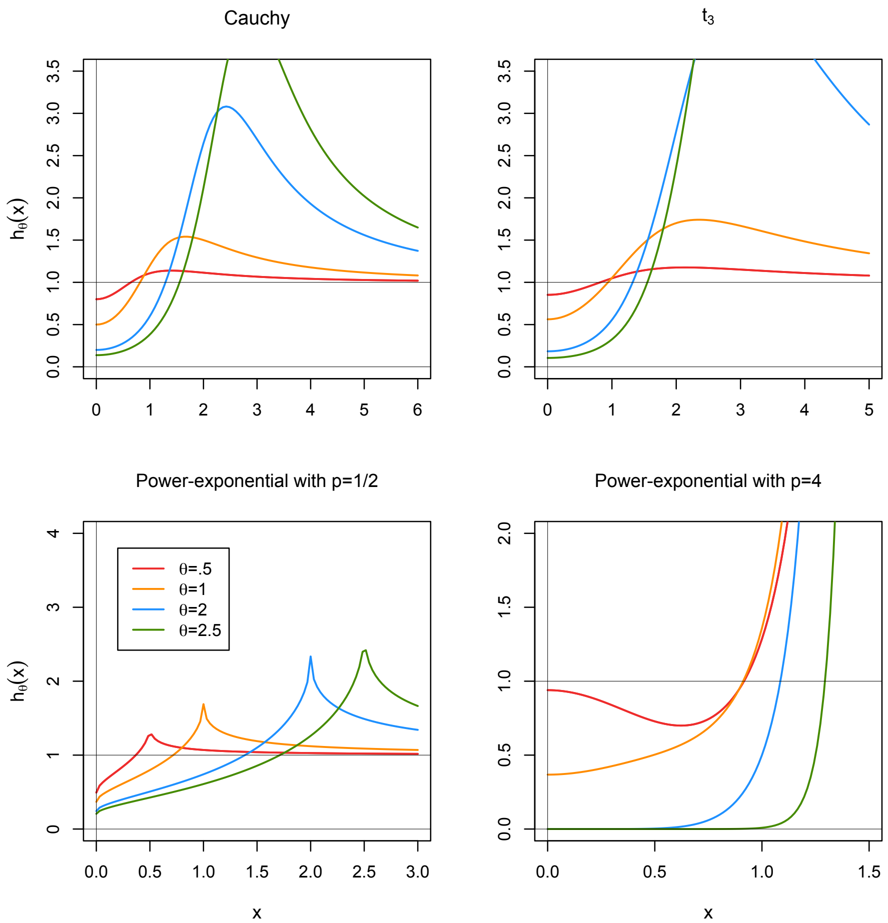

Fig. 4 Plots of the SLR functions $x \mapsto h_\theta(x)$ over positive values of x for $\theta = .5$, $\theta = 1$, $\theta = 2$ and $\theta = 2.5$, when f_0 is the Cauchy density, the density of the t distribution with three degrees of freedom, and the power-exponential densities with $p = 1/2$ and $p = 4$.

Proposition 4 *Let θ_0 be such that $2\theta_0 - \theta \in \Theta$ for any $\theta \in \Theta$. Then, the model described by the densities in (15) is symmetric about θ_0 if and only if (i) the distribution of T under P_{θ_0} is symmetric about its mean $\mathrm{E}_{\theta_0}[T]$ and (ii) $\eta(2\theta_0 - \theta) - \eta(\theta_0) = -(\eta(\theta) - \eta(\theta_0))$ for any θ.*

Our main result in this section is then the following.

Theorem 3 *Consider an exponential model that is symmetric about θ_0 and fix $\alpha \in (0, 1)$. Then, (i) there exist $\gamma_\alpha \in [0, 1]$ and $s_\alpha \geq 0$ such that the test* *defined by*

$$\phi_\alpha(x) = \begin{cases} 1 & \textit{if } |T(x) - \mathrm{E}_{\theta_0}[T]| > s_\alpha \\ \gamma_\alpha & \textit{if } |T(x) - \mathrm{E}_{\theta_0}[T]| = s_\alpha \\ 0 & \textit{otherwise} \end{cases} \tag{16}$$

satisfies $\mathrm{E}_{\theta_0}[\phi_\alpha] = \alpha$*; (ii) the power function* $\theta \mapsto \mathrm{E}_\theta[\phi_\alpha]$ *is symmetric about* θ_0*; (iii)* ϕ_α *is UMPS at level* α *when testing* $\mathcal{H}_0 : \theta = \theta_0$ *against* $\mathcal{H}_1 : \theta \neq \theta_0$.

Proof of Theorem 3. (i) Letting $t \mapsto F^S_{\theta_0}(t) := \mathrm{P}^S_{\theta_0}[(-\infty, t]]$ be the cumulative distribution function of $S := T - \mathrm{E}_{\theta_0}[T]$ when X has distribution P_{θ_0}, define

$$s_\alpha := \inf\{s \in \mathbb{R} : F^S_{\theta_0}(s) \geq 1 - (\alpha/2)\}$$

and

$$\gamma_\alpha := \begin{cases} (F^S_{\theta_0}(s_\alpha) - (1 - (\alpha/2)))/\mathrm{P}^S_{\theta_0}[\{s_\alpha\}] & \text{if } \mathrm{P}^S_{\theta_0}[\{s_\alpha\}] > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Since $\mathrm{P}^S_{\theta_0}$ is symmetric, the test in (16) then satisfies

$$\begin{aligned} \mathrm{E}_{\theta_0}[\phi_\alpha] &= \mathrm{P}^S_{\theta_0}[(-\infty, -s_\alpha)] + \mathrm{P}^S_{\theta_0}[(s_\alpha, \infty)] + \gamma_\alpha \mathrm{P}^S_{\theta_0}[\{-s_\alpha, s_\alpha\}] \\ &= 2\mathrm{P}^S_{\theta_0}[(s_\alpha, \infty)] + 2\gamma_\alpha \mathrm{P}^S_{\theta_0}[\{s_\alpha\}] \\ &= 2\big(1 - F^S_{\theta_0}(s_\alpha) + \gamma_\alpha \mathrm{P}^S_{\theta_0}[\{s_\alpha\}]\big) \\ &= \alpha, \end{aligned}$$

which shows the result. (ii) Since $\phi_\alpha = g_\alpha(S)$ for some function g_α satisfying $g_\alpha(-s) = g_\alpha(s)$ for any s, the symmetry assumption on the model entails that, for any θ, we have that ϕ_α has the same distribution under $\mathrm{P}_{2\theta-\theta_0}$ as under P_θ. This implies in particular that $\mathrm{E}_{2\theta_0-\theta}[\phi] = \mathrm{E}_\theta[\phi]$ for any θ. (iii) Fix $\theta_1 \in \Theta \setminus \{\theta_0\}$ arbitrarily. Let us show that there exist $k_1, k_2 \geq 0$ such that the test ϕ_{k_1,k_2} defined by

$$\phi_{k_1,k_2}(x) = \begin{cases} 1 & \text{if } g(x) > 0 \\ \gamma_\alpha & \text{if } g(x) = 0 \\ 0 & \text{if } g(x) < 0, \end{cases}$$

with

$$g(x) := \frac{1}{f_{\theta_0}(x)} \big\{ f_{\theta_1}(x) - k_1 f_{\theta_0}(x) - k_2(f_{\theta_1}(x) - f_{2\theta_0-\theta_1}(x)) \big\},$$

coincides with ϕ_α. Using the expression of the densities in (15), symmetry of the exponential model about θ_0, and Proposition 4, we have

$$\begin{aligned} g(x) &= \frac{(1-k_2)C(\theta_1)}{C(\theta_0)} e^{(\eta(\theta_1)-\eta(\theta_0))T(x)} + \frac{k_2 C(2\theta_0 - \theta_1)}{C(\theta_0)} e^{(\eta(2\theta_0-\theta_1)-\eta(\theta_0))T(x)} - k_1 \\ &= \frac{r(1-k_2)C(\theta_1)}{C(\theta_0)} e^{a(T(x)-\mathrm{E}_{\theta_0}[T])} + \frac{k_2 C(2\theta_0 - \theta_1)}{rC(\theta_0)} e^{-a(T(x)-\mathrm{E}_{\theta_0}[T])} - k_1, \end{aligned}$$

where we let $a := \eta(\theta_1) - \eta(\theta_0)$ and $r := e^{(\eta(\theta_1)-\eta(\theta_0))\mathrm{E}_{\theta_0}[T]}$. For

$$k_2 = \frac{r^2 C(\theta_1)}{C(2\theta_0 - \theta_1) + r^2 C(\theta_1)},$$

we thus have

$$\begin{aligned} g(x) &= \frac{rC(\theta_1)C(2\theta_0-\theta_1)}{C(\theta_0)(C(2\theta_0-\theta_1)+r^2C(\theta_1))}\big(e^{a(T(x)-\mathrm{E}_{\theta_0}[T])} + e^{-a(T(x)-\mathrm{E}_{\theta_0}[T])}\big) - k_1 \\ &=: \lambda\big(e^{a(T(x)-\mathrm{E}_{\theta_0}[T])} + e^{-a(T(x)-\mathrm{E}_{\theta_0}[T])}\big) - k_1, \end{aligned}$$

where λ is a fixed positive real number. Since $a \neq 0$ (having $a = 0$ would violate injectivity of the parametrization), there exists a (unique) nonnegative real number k_1 such that the test ϕ_{k_1,k_2} coincides with ϕ_α, hence satisfies $\mathrm{E}_{\theta_0}[\phi_{k_1,k_2}] = \alpha$ and $\mathrm{E}_{\theta_1}[\phi_{k_1,k_2}] - \mathrm{E}_{2\theta_0-\theta_1}[\phi_{k_1,k_2}] = 0$. Since $k_1, k_2 \geq 0$, Theorem 3.6.1(iii) in Lehmann and Romano (2022) entails that $\phi_\alpha = \phi_{k_1,k_2}$ is most powerful when testing $\mathcal{H}_0 : \theta = \theta_0$ against $\mathcal{H}_1 : \theta = \theta_1$ in the class of tests satisfying

$$\mathrm{E}_{\theta_0}[\phi] = \int \phi(x) f_{\theta_0}(x)\, d\mu(x) \leq \alpha \tag{17}$$

and

$$\mathrm{E}_{\theta_1}[\phi] - \mathrm{E}_{2\theta_0-\theta_1}[\phi] = \int \phi(x)(f_{\theta_1}(x) - f_{2\theta_0-\theta_1}(x))\, d\mu(x) \leq 0, \tag{18}$$

hence also in the smaller class of level-α tests whose power function is symmetric about θ_0. Since ϕ_α does not depend on θ_1, the result follows. □

Let us consider two examples that enter the exponential framework considered here. If $X = (X_1, \ldots, X_n)$ collects mutually independent random variables from the Bernoulli distribution with parameter p, then X admits the density, with respect to the counting measure on $\{0,1\}^n$,

$$f_p(x) = p^{\sum_{i=1}^n x_i}(1-p)^{n-\sum_{i=1}^n x_i} = C(p)\exp(\eta(p)T(x)),$$

with $C(p) = (1-p)^n$, $\eta(p) = \log(p/(1-p))$ and $T(x) = \sum_{i=1}^n x_i$. Using Proposition 4, this exponential family is seen to be symmetric with respect to $p_0 = 1/2$. For the problem of testing $\mathcal{H}_0 : p = 1/2$ against $\mathcal{H}_1 : p \neq 1/2$, Theorem 3 thus implies that the two-sided test defined by

$$\phi_\alpha(x) = \begin{cases} 1 & \text{if } |(\sum_{i=1}^n x_i) - \frac{n}{2}| > s_\alpha \\ \gamma_\alpha & \text{if } |(\sum_{i=1}^n x_i) - \frac{n}{2}| = s_\alpha \\ 0 & \text{if } |(\sum_{i=1}^n x_i) - \frac{n}{2}| < s_\alpha, \end{cases}$$

where $s_\alpha \geq 0$ and $\gamma_\alpha \in [0,1]$ are chosen such that $\mathrm{E}_{p_0}[\phi_\alpha] = \alpha$ (based on the fact that $T \sim \mathrm{Bin}(n, 1/2)$ under $p = p_0$), is UMPS at level α. Now, if $X = (X_1, \ldots, X_n)$ rather collects mutually independent random variables from the Gaussian distribution with mean μ and fixed variance σ_0^2, then the density of X with respect to the Lebesgue measure on $\mathbb{R}^n$ is

$$f_\mu(x) = \left(\frac{1}{2\pi\sigma_0^2}\right)^{\frac{n}{2}} \exp\left(-\frac{1}{2\sigma_0^2}\sum_{i=1}^n (x_i-\mu)^2\right) = C(\mu)h(x)\exp(\eta(\mu)T(x)),$$

with

$$C(\mu) = \left(\frac{1}{2\pi\sigma_0^2}\right)^{\frac{n}{2}} \exp\left(-\frac{n\mu^2}{2\sigma_0^2}\right), \quad h(x) = \exp\left(-\frac{1}{2\sigma_0^2}\sum_{i=1}^n x_i^2\right), \quad \eta(\mu) = \frac{\mu}{\sigma_0^2},$$

and with the same statistic T as in the previous example. Still from Proposition 4, this exponential model is symmetric with respect to any $\mu_0 \in \mathbb{R}$, and, denoting as Φ the cumulative distribution function of the standard normal, Theorem 3 entails that the usual two-sided test ϕ_α rejecting $\mathcal{H}_0 : \mu = \mu_0$ in favour of $\mathcal{H}_1 : \mu \neq \mu_0$ when $|(\frac{1}{n}\sum_{i=1}^n x_i) - \mu_0| > \sigma_0 \Phi^{-1}(1-\frac{\alpha}{2})/\sqrt{n}$ is UMPS at level α.

5 Wrap up and final comments

Hypothesis testing problems allowing for UMP tests at a given significance level are exceptions rather than the rule. When such UMP tests are indeed unavailable, the way out consists in adopting a statistical principle, that restricts the class of competing tests to a collection of tests meeting a natural, desirable, property. Maybe the most common statistical principle in hypothesis testing is the *unbiasedness principle*, which leads to restricting to the class of unbiased tests. In this work, we introduced a *symmetry principle*, that is at least as natural as the unbiasedness principle when the statistical model at hand is itself symmetric, as it is the case for the location model with a base symmetric density. We exhibited numerous specific hypothesis testing problems in which existence of a test that is UMP among unbiased tests remains an open problem yet in which we could show that the natural symmetric test is UMP among tests having a symmetric power function. When we could not build such a UMP symmetric test, we showed that a UMP test may exist in the smaller class of tests whose power function is symmetric and U-shaped, which is another original statistical principle, the *symmetry U-shaped principle*, say.

Another, less common, yet classical, statistical principle is the *invariance principle*, which leads to restricting to tests that invariant under groups of transformations leaving both the null hypothesis and alternative hypothesis invariant; see, e.g., Section 6.1 in Lehmann and Romano (2022). The symmetry principle we introduced is much closer in spirit to the invariance principle than to the unbiasedness principle, yet there are key differences between these principles. Classically, one of the main motivations to adopt the invariance principle is that, as soon as the group of transformations is large enough to generate the null hypothesis at hand, invariant tests are distribution-free under the null hypothesis, which makes designing a critical value as straightforward as if the original problem would have involved a simple null hypothesis. In the testing problems we considered in this work, however, this is obviously irrelevant since the original null hypothesis $\mathcal{H}_0 : \theta = \theta_0$ was already simple, hence does not need any reduction through invariance. Now, standard invariance arguments imply that invariant tests are distribution-free along the orbits of the induced group of transformations, which, for the problems we considered, leads to tests with symmetric power functions. To be more specific, let us focus on the last example we tackled in this work, namely the problem of testing $\mathcal{H}_0 : \mu = \mu_0$ against $\mathcal{H}_0 : \mu \neq \mu_0$ based on observations $X_1, \ldots, X_n$ that are i.i.d. normal with mean μ and given variance σ_0^2. The null and alternative hypotheses are invariant under the group of transformations made of the identity mapping and of the mapping reflecting each observation with respect to the null value μ_0. The induced group of transformations acts on $\{\mu : \mu \in \mathbb{R}\}$ and similarly contains the identity mapping and the reflection with respect to μ_0. Invariant tests being distribution-free along the orbits of this induced group, such tests will by construction have a symmetric power function. In this sense, the invariance principle imposes a stronger restriction than the proposed symmetry principle, which is an argument in favour of the symmetry principle (the same conclusion can be reached in the Bernoulli example from the previous section by considering the group of transformations collecting the identity mapping and the mapping $(x_1, \ldots, x_n) \mapsto (1 - x_1, \ldots, 1 - x_n)$).

The only multi-observation problems we tackled in this work involve exponential models. While this may seem to be a limitation, it might be so that, under mild regularity conditions, existence of a UMPS test at a given significance level for any sample size n requires the exponential paradigm. This is actually the case already for existence of one-sided UMP tests; see Pfanzagl (1968). It might be explored in future research whether this is also the case for UMPS tests, although it seems extremely challenging to obtain a formal result in this direction. Finally, we note that existence of UMPS tests and UMPSU tests in this work was established by applying the generalized Neyman–Pearson fundamental lemma in an original way. It would be interesting to see whether the same strategy can be exploited to establish existence of optimal tests relative to other, original, statistical principles.

6 Proofs of auxiliary results

We first show that the SLR functions associated with the power-exponential densities with $p \in (1, 2]$ satisfy the monotonicity condition from Theorem 1 (Proposition 5 below). This requires the following preliminary result.

Lemma 1 *Fix $\theta > 0$. Let $r, s : [0, \infty) \to \mathbb{R}$ be continuous functions such that $r(x) > s(x)$ for any $x > 0$. Assume further that r and s are differentiable on $(0, \infty) \setminus \{\theta\}$ and that $r'(x) \geq -s'(x) > 0$ for any $x \in (0, \infty) \setminus \{\theta\}$. Then the function defined by*

$$h(x) = \frac{1}{2}(e^{r(x)} + e^{s(x)})$$

is strictly increasing over $[0, \infty)$.

Proof of Lemma 1. For any $x \in (0, \infty) \setminus \{\theta\}$,

$$2h'(x) = r'(x)e^{r(x)} - (-s'(x))e^{s(x)} \geq r'(x)(e^{r(x)} - e^{s(x)}) > 0,$$

so that h is strictly increasing on $(0, \theta)$ and on (θ, ∞). Since h is continuous on $[0, \infty)$, it is then strictly increasing on $[0, \infty)$. □

Proposition 5 *Fix* $\theta > 0$ *and* $p \in (1, 2]$. *Then the SLR function* h_θ *in (2) is strictly increasing over* $[0, \infty)$.

Proof of Proposition 5. We prove the result by applying Lemma 1 with

$$r(x) := |x|^p - |x - \theta|^p \quad \text{and} \quad s(x) := |x|^p - |x + \theta|^p,$$

that obviously define continuous functions on $[0, \infty)$. For any $x > 0$, we have $|x + \theta| > |x - \theta|$, hence $r(x) > s(x)$. These functions are differentiable at any $x \in (0, \infty) \setminus \{\theta\}$, with

$$r'(x) = px^{p-1} - p|x - \theta|^{p-1}\mathrm{Sign}(x - \theta) \quad \text{and} \quad s'(x) = px^{p-1} - p(x + \theta)^{p-1}.$$

Since one clearly has $-s'(x) > 0$ for any $x \in (0, \infty) \setminus \{\theta\}$, it only remains to show that $r'(x) \geq -s'(x)$ at any such x, that is, it remains to show that

$$(x + \theta)^{p-1} + |x - \theta|^{p-1}\mathrm{Sign}(x - \theta) \leq 2x^{p-1} \tag{19}$$

for any $x \in (0, \infty) \setminus \{\theta\}$. Now, for $x \in (\theta, \infty)$, this readily follows from the fact that $t \mapsto t^{p-1}$ is concave on $[x - \theta, x + \theta]$. For $x \in (0, \theta)$, the C_r-inequality (see, e.g., Bilodeau and Brenner (1999), page 33) provides

$$(\theta + x)^{p-1} = (\theta - x + 2x)^{p-1} \leq (\theta - x)^{p-1} + (2x)^{p-1} \leq (\theta - x)^{p-1} + 2x^{p-1},$$

which proves (19), hence establishes the result. □

We now prove Proposition 3.

Proof of Proposition 3. (i) Since the power function of ϕ_α is given by

$$\begin{aligned}\theta \mapsto \mathrm{E}_\theta[\phi_\alpha] = \mathrm{P}_\theta[|X - \theta_0| > s_\alpha] &= 1 - \int_{\theta_0 - s_\alpha}^{\theta_0 + s_\alpha} f_0(x - \theta)\,dx \\ = 1 - \int_{-s_\alpha}^{s_\alpha} f_0(z - (\theta - \theta_0))\,dz &= 1 - \int_{-s_\alpha}^{s_\alpha} f_0(z - |\theta - \theta_0|)\,dz,\end{aligned} \tag{20}$$

the power function is always symmetric about θ_0. Now, denoting as F_0 is the cumulative distribution function associated with the density f_0, unimodality of f_0 ensures that

$$c \mapsto g(c) := \int_{-s_\alpha}^{s_\alpha} f_0(z - c)\,dz = F_0(s_\alpha - c) - F_0(-s_\alpha - c)$$

satisfies $g'(c) = f_0(-s_\alpha - c) - f_0(s_\alpha - c) \leq 0$ for any $c > 0$. It follows that g is a monotone non-increasing function over $\mathbb{R}^+$, hence that the power function of ϕ_α in (20) is a monotone non-decreasing function of $|\theta - \theta_0|$. It follows that this power function is symmetric about θ_0 and U-shaped.

(ii) Assume that f_0 is not unimodal, so that there exist $z_2 > z_1 \geq 0$ such that $f_0(z_2) > f_0(z_1)$. From continuity, the mapping $\alpha \mapsto s_\alpha$ from $(0,1)$ to $(0,\infty)$ is surjective, so that there exists $\alpha_0 \in (0,1)$ such that $s_{\alpha_0} = (z_2 - z_1)/2$. Thus, at $c = z_1 + s_{\alpha_0}$, we have

$$\begin{aligned} g'(c) &= f_0(-s_{\alpha_0} - c) - f_0(s_{\alpha_0} - c) \\ &= f_0(s_{\alpha_0} + c) - f_0(-s_{\alpha_0} + c) \\ &= f_0(z_2) - f_0(z_1) > 0. \end{aligned}$$

Continuity of f_0 then implies that g' is strictly positive in a neighbourhood $\mathcal{N}$ of c, which, in view of the computations in Part (i) of the proof, entails that the power function $\theta \mapsto \mathrm{E}_\theta[\phi_{\alpha_0}]$ is a monotone strictly decreasing function of $|\theta - \theta_0|$ for any value of $|\theta - \theta_0| \in \mathcal{N}$. Consequently, this power function is not U-shaped. □

We turn to the proof of Proposition 4, which requires the following preliminary result.

Lemma 2 *There is no $\theta \in \Theta$ such that T is P_θ-almost surely constant.*

Proof of Lemma 2. Ad absurdum, assume that there exist $\theta \in \Theta$ and a possible value t of T such that $\mathrm{P}_\theta[T = t] = 1$. Fix an arbitrary $\theta_1 \in \Theta \setminus \{\theta\}$. For any $A \in \mathcal{A}$, we have

$$\begin{aligned} \mathrm{P}_{\theta_1}[A] &= \int_A f_{\theta_1}(x)\, d\mu(x) = \int_A \frac{f_{\theta_1}(x)}{f_\theta(x)}\, d\mathrm{P}_\theta(x) \\ &= \frac{C(\theta_1)}{C(\theta)} \int_A \exp((\eta(\theta_1) - \eta(\theta))T(x))\, d\mathrm{P}_\theta(x) \\ &= \frac{C(\theta_1)}{C(\theta)} \exp((\eta(\theta_1) - \eta(\theta))t)\mathrm{P}_\theta[A]. \end{aligned}$$

Taking $A = \mathcal{X}$ entails that

$$\frac{C(\theta_1)}{C(\theta)} \exp((\eta(\theta_1) - \eta(\theta))t) = 1,$$

which in turn implies that $\mathrm{P}_{\theta_1} = \mathrm{P}_\theta$, a contradiction. □

Proof of Proposition 4. Before proving the result, we first make the following general considerations. Rewrite the densities in (15) as

$$\begin{aligned} f_\theta(x) &= C(\theta)\exp(\eta(\theta)\mathrm{E}_{\theta_0}[T])h(x)\exp(\eta(\theta)S(x)) \\ &=: D(\theta)h(x)\exp(\eta(\theta)S(x)). \end{aligned}$$

Changing the dominating measure to the measure ξ defined through $\xi(A) = \int_A h(x)\,d\nu(x)$ yields the densities

$$\frac{d\mathrm{P}_\theta}{d\xi}(x) = \frac{\frac{d\mathrm{P}_\theta}{d\nu}(x)}{\frac{d\xi}{d\nu}(x)} = D(\theta)\exp(\eta(\theta)S(x)).$$

Letting $(\mathcal{S}, \mathcal{B})$ be the measure space associated with $S = T - \mathrm{E}_{\theta_0}[T]$, we then have, for any $B \in \mathcal{B}$,

$$\begin{aligned} \mathrm{P}^S_\theta[B] = \mathrm{P}_\theta[S^{-1}(B)] &= \int_{S^{-1}(B)} D(\theta)\exp(\eta(\theta)S(x))\,d\xi(x) \\ &= \int_B D(\theta)\exp(\eta(\theta)s)\,d\xi^S(s), \end{aligned}$$

where ξ^S is the measure defined through $\nu^S(B) = \nu(S^{-1}(B))$. It follows that $f^S_\theta(s) := D(\theta)\exp(\eta(\theta)s)$ is a version of the Radon–Nykodim derivative of P^S_θ with respect to ξ^S. Therefore, for any $B \in \mathcal{B}$, we have

$$\begin{aligned} \mathrm{P}^S_\theta[B] &= \int_B \frac{d\mathrm{P}^S_\theta/d\xi^S(s)}{d\mathrm{P}^S_{\theta_0}/d\xi^S(s)}\,d\mathrm{P}^S_{\theta_0}(s) \\ &= \frac{D(\theta)}{D(\theta_0)}\int_B \exp((\eta(\theta)-\eta(\theta_0))s)\,d\mathrm{P}^S_{\theta_0}(s), \end{aligned}$$

so that

$$s \mapsto \frac{D(\theta)}{D(\theta_0)}\exp((\eta(\theta)-\eta(\theta_0))s) = \frac{\exp((\eta(\theta)-\eta(\theta_0))s)}{\int_{\mathcal{S}}\exp((\eta(\theta)-\eta(\theta_0))s)\,d\mathrm{P}^S_{\theta_0}(s)}$$

is a version of the Radon–Nykodim derivative of P^S_θ with respect to $\mathrm{P}^S_{\theta_0}$. We can now prove the result.

($\Rightarrow$) Assume that the exponential model is symmetric about θ_0. By assumption, the distribution of T under P_{θ_0} is symmetric with respect to its mean $\mathrm{E}_{\theta_0}[T]$, so that we only need to show that $\eta(2\theta_0-\theta)-\eta(\theta_0) = -(\eta(\theta)-\eta(\theta_0))$ for any θ. To do so, fix $\theta \neq \theta_0$ arbitrarily (the claim is trivial for $\theta = \theta_0$). For any $B \in \mathcal{B}$, the symmetry assumption implies that

$$\mathrm{P}^S_{2\theta_0-\theta}[B] = \frac{\int_B \exp((\eta(2\theta_0-\theta)-\eta(\theta_0))s)\,d\mathrm{P}^S_{\theta_0}(s)}{\int_{\mathcal{S}}\exp((\eta(2\theta_0-\theta)-\eta(\theta_0))s)\,d\mathrm{P}^S_{\theta_0}(s)}$$

is equal to

$$\begin{aligned}
\mathrm{P}_\theta^S[-B] &= \frac{\int_{-B} \exp((\eta(\theta)-\eta(\theta_0))s)\, d\mathrm{P}_{\theta_0}^S(s)}{\int_{\mathcal{S}} \exp((\eta(\theta)-\eta(\theta_0))s)\, d\mathrm{P}_{\theta_0}^S(s)} \\
&= \frac{\int_{B} \exp(-(\eta(\theta)-\eta(\theta_0))s)\, d\mathrm{P}_{\theta_0}^S(s)}{\int_{\mathcal{S}} \exp(-\eta(\theta)-\eta(\theta_0))s)\, d\mathrm{P}_{\theta_0}^S(s)},
\end{aligned}$$

where we used the fact $\mathrm{P}_{\theta_0}^S$ is symmetric by assumption. Thus,

$$s \mapsto \frac{\exp((\eta(2\theta_0-\theta)-\eta(\theta_0))s)}{\int_{\mathcal{S}} \exp((\eta(2\theta_0-\theta)-\eta(\theta_0))s)\, d\mathrm{P}_{\theta_0}^S(s)} \tag{21}$$

and

$$s \mapsto \frac{\exp(-(\eta(\theta)-\eta(\theta_0))s)}{\int_{\mathcal{S}} \exp(-(\eta(\theta)-\eta(\theta_0))s)\, d\mathrm{P}_{\theta_0}^S(s)} \tag{22}$$

are versions of the Radon–Nykodim derivative of $\mathrm{P}_{2\theta-\theta_0}^S$ with respect to $\mathrm{P}_{\theta_0}^S$, hence coincide $\mathrm{P}_{\theta_0}^S$-almost everywhere. Assume that $\eta(2\theta_0-\theta)-\eta(\theta_0) \neq -(\eta(\theta)-\eta(\theta_0))$. Since injectivity of the parametrization ensures that both $\eta(\theta)-\eta(\theta_0)$ and $\eta(2\theta_0-\theta)-\eta(\theta_0)$ are non-zero, the Radon–Nykodim derivatives in (21)–(22) can be equal for at most one value of s, which implies that $\mathrm{P}_{\theta_0}^S$ is a Dirac probability measure. Since this contradicts Lemma 2, we must then have that $\eta(2\theta_0-\theta)-\eta(\theta_0) = -(\eta(\theta)-\eta(\theta_0))$, as was to be shown.

($\Leftarrow$) Assume that the distribution of T under P_{θ_0} is symmetric with respect to its mean $\mathrm{E}_{\theta_0}[T]$ and that $\eta(2\theta_0-\theta)-\eta(\theta_0) = -(\eta(\theta)-\eta(\theta_0))$ for any θ. Fix $\theta \neq \theta_0$ arbitrarily. Using the symmetry assumption on $\mathrm{P}_{\theta_0}^S$, we have

$$\begin{aligned}
\mathrm{P}_{2\theta-\theta_0}^S[B] &= \frac{\int_{B} \exp((\eta(2\theta-\theta_0)-\eta(\theta_0))s)\, d\mathrm{P}_{\theta_0}^S(s)}{\int_{\mathcal{S}} \exp((\eta(2\theta-\theta_0)-\eta(\theta_0))s)\, d\mathrm{P}_{\theta_0}^S(s)} \\
&= \frac{\int_{B} \exp(-(\eta(\theta)-\eta(\theta_0))s)\, d\mathrm{P}_{\theta_0}^S(s)}{\int_{\mathcal{S}} \exp(-(\eta(\theta)-\eta(\theta_0))s)\, d\mathrm{P}_{\theta_0}^S(s)} \\
&= \frac{\int_{-B} \exp((\eta(\theta)-\eta(\theta_0))s)\, d\mathrm{P}_{\theta_0}^S(s)}{\int_{\mathcal{S}} \exp(-(\eta(\theta)-\eta(\theta_0))s)\, d\mathrm{P}_{\theta_0}^S(s)} \\
&= \mathrm{P}_\theta^S[-B]
\end{aligned}$$

for any $B \in \mathcal{B}$. This shows that the distribution of S under $\mathrm{P}_{2\theta_0-\theta}$ is the same as the distribution of $-S$ under P_θ, which establishes the result. □

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References

Bilodeau, M., & Brenner, D. (1999). *Theory of Multivariate Statistics*. New York: Springer.
Lehmann, E.L., & Romano, J.P. (2022). *Testing Statistical Hypotheses*. Cham: Springer.
Pfanzagl, J. (1968). A characterization of the one parameter exponential family by existence of uniformly most powerful tests. *Sankhyā A*, 30, 147–156.
Shao, J. (2003). *Mathematical Statistics*. New York: Springer.