On the robustness of spatial quantiles

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Abstract. Spatial quantiles are among the most successful concepts of multivariate quantiles. In particular, they are essentially the only quantiles that can be computed in high dimensions. There has been an intense research activity to study spatial quantiles in the last two decades, yet surprisingly little is known about their robustness properties. In the present work, we carefully study the breakdown point of spatial quantiles. We offer three approaches, that show diverse distinctive advantages. The first approach is a constructive one: it is conceptually simple and allows us to derive the finite-sample breakdown point of spatial quantiles. While the second approach is not constructive and does not identify the global breakdown point of spatial quantiles, it provides an upper bound on the breakdown point under contamination in any fixed direction. It also allows us to determine the breakdown point of L_p-spatial quantiles for any $p > 1$. Last but not least, the third approach characterizes precisely when breakdown occurs under any given contamination scheme, hence provides the breakdown points associated with very diverse contamination scenarios. Quite nicely, this last approach further covers cases where the contamination and/or the ground probability measures are continuous distributions. An intriguing corollary of our results states that, in high dimensions, the "practical" breakdown point exceeds the theoretical one. Throughout, our theoretical results are illustrated through numerical exercises. Part of our results cover infinite-dimensional Hilbert spaces as well.

Résumé. Les quantiles spatiaux comptent parmi les concepts de quantiles multivariés qui ont rencontré le plus grand succès. En particulier, ces quantiles sont essentiellement les seuls qui peuvent être calculés en grande dimension. Ils ont fait l'objet d'une activité de recherche intense au cours des deux dernières décennies, mais, de façon surprenante, on sait très peu au sujet de leur robustesse. Dans ce travail, nous étudions en détail le point de rupture des quantiles spatiaux. Nous adoptons trois approches, qui ont chacune des avantages propres. La première approche est constructive: elle est conceptuellement simple et nous permet d'obtenir le point de rupture exact des quantiles spatiaux. Si la seconde approche n'est pas constructive et ne permet pas d'identifier le point de rupture global des quantiles spatiaux, elle fournit une borne supérieure sur le point de rupture associé à des contaminations dans une direction fixée. Elle nous permet par ailleurs de déterminer le point de rupture des quantiles spatiaux L_p pour tout $p > 1$. Enfin et surtout, la troisième approche caractérise précisément quand la rupture survient sous un quelconque schéma de contamination fixé, ce qui fournit les points de rupture associés à des scenarios de contamination très divers. De façon intéressante, cette dernière approche couvre de plus les cas où la contamination et/ou la mesure de probabilité de référence sont des distributions continues. Un corollaire surprenant de nos résultats montre que le point de rupture "pratique" en grande dimension excède le point de rupture théorique. Tout du long, nos résultats théoriques sont illustrés par des exercices numériques. Une partie de nos résultats couvrent également les espaces de Hilbert de dimension infinie.

Keywords: Breakdown point, M-quantiles, Multivariate quantiles, Robustness, Spatial quantiles

1. Introduction

The problem of defining multivariate quantiles, i.e. quantiles for probability measures over the d-dimensional Euclidean space, has attracted much attention in the last two decades. One of the most celebrated solutions is the concept of *spatial* (or *geometric*) *quantiles*, that was introduced in [7] and [11]. Spatial quantiles are defined as follows. For a probability measure P over $\mathbb{R}^d$, a spatial quantile of order α in direction u, with $\alpha \in [0,1)$ and u in the unit sphere $\mathcal{S}^{d-1} = \{x \in \mathbb{R}^d : \|x\|^2 = x'x = 1\}$ of $\mathbb{R}^d$, is defined as a minimizer of

$$\mu \mapsto M_{\alpha,u}(\mu; P) = \int_{\mathbb{R}^d} (\|x-\mu\| - \|x\|)\, dP(x) - \alpha u'\mu \tag{1.1}$$

over $\mathbb{R}^d$. While existence is guaranteed, uniqueness is not. To unambiguously define spatial quantiles in this work, we *choose* to define *the* spatial quantile of order α in direction u for P, denoted as $\mu_{\alpha,u}(P)$ or simply $\mu_{\alpha,u}$, as the barycentre of the collection of minimizers of the objective function (1.1) (since this function is convex, the collection of minimizers is a convex subset of $\mathbb{R}^d$ and its barycentre is itself a minimizer). When spatial quantiles are not uniquely defined, the robustness results we will obtain below depend on this choice, as is already the case in dimension $d = 1$.

Spatial quantiles belong to the family of "center-outward" quantiles concepts: the innermost quantile is associated with $\alpha = 0$ (and an arbitrary direction u) and is known as the *spatial median* or L_1*-median* (see [4], [22], [24]), whereas, typically, the larger α, the less central the spatial quantile $\mu_{\alpha,u}$ in direction u (the term "typically" is needed in view of the results from [26]). Therefore, a natural outlyingness measure $O(x; P)$ for a given location x in $\mathbb{R}^d$ with respect to P is the order α of the spatial quantile $\mu_{\alpha,u}$ equal to x (under mild assumptions, it can be shown that the function from the open unit ball in $\mathbb{R}^d$ to $\mathbb{R}^d$ that maps αu to $\mu_{\alpha,u}$ is one-to-one, which justifies this definition).

As usual, sample quantiles are obtained by considering the population concept above with the empirical probability measure associated with the sample at hand. More precisely, for a given d-variate sample $x_1, \ldots, x_n$, the spatial quantile of order α in direction u is defined as the barycentre, $\mu_{\alpha,u}(x_1, \ldots, x_n)$ say, of the set of minimizers of

$$\mu \mapsto M_{\alpha,u}(\mu; x_1, \ldots, x_n) = \frac{1}{n}\sum_{i=1}^{n}(\|x_i - \mu\| - \|x_i\|) - \alpha u'\mu \tag{1.2}$$

over $\mathbb{R}^d$ (while the term $\|x\|$ in (1.1) is needed to guarantee that the objective function is well-defined without any moment assumption, the term $\|x_i\|$ in (1.2) can of course be safely dropped in the empirical case). Note that the convention that is adopted here to identify a unique representative of the set of possible quantiles is the one that is traditionally used when considering the univariate median with an even sample size. Spatial quantiles have been much investigated. Their consistency and asymptotic distribution were studied in [7], and Bahadur representation results were obtained in [17] and [18]; see also [33]. In the population case, extreme spatial quantiles (i.e., those obtained with α close to one) were studied in [12] and [13], with a motivation arising from extreme value theory; part of these results were extented to the empirical case in [27]. Spatial quantiles are minimizers of a convex objective function and can be computed very efficiently, even in high dimensions; see [25] and [31]. New properties of spatial quantiles have recently been identified in [28] when using these quantiles in a general robustification procedure.

Spatial quantiles were generalized in various directions. They were used in a regression context to define a multiple-output version of quantile regression; see, e.g., [6] and [8]. The definition of spatial quantiles naturally also makes sense in a general Hilbert space, which explains that these quantiles were also investigated in infinite-dimensional spaces, in particular in a functional data analysis context; see [5], [9], [29], or [30]. Spatial quantiles have also been recently defined on unit spheres ([20]), where they provide alternatives to spherical-cap quantiles ([21]) or to quantiles based on measure transportation ideas ([15]). Coming back to the d-dimensional Euclidean case, the construction of spatial quantiles inspired the definition of the spatial expectiles in [16], and, more generally, the definition of general ρ-quantiles in [19]. To be more specific, for a non-decreasing loss function $\rho : [0, \infty) \to [0, \infty)$, a spatial ρ-quantile of order α in direction u is defined as a minimizer of

$$\mu \mapsto M_{\alpha,u,\rho}(\mu; P) = \int_{\mathbb{R}^d} (H_{\alpha,u,\rho}(x - \mu) - H_{\alpha,u,\rho}(x))\, dP(x), \tag{1.3}$$

where we let

$$H_{\alpha,u,\rho}(z) = \rho(\|z\|)\left(1 + \alpha\frac{u'z}{\|z\|}\right)\mathbb{I}[z \neq 0];$$

here, $\mathbb{I}[A]$ denotes the indicator function associated with condition A. The L_1 loss function defined by $\rho(t) = t$ provides the spatial quantiles above, whereas the L_2 loss function defined by $\rho(t) = t^2$ provides the spatial expectiles from [16]. More generally, spatial L_p-quantiles are obtained with $\rho(t) = t^p$, and even more general loss functions yield a spatial concept of multivariate M-quantiles; see [3]. We refer to [19] for a systematic investigation of spatial ρ-quantiles.

Of course, it is expected that spatial quantiles, that are L_1 concepts, will be more robust that their L_p-counterparts above; in line with this, while the innermost spatial quantile (obtained with $\alpha = 0$) is a multivariate median, the innermost spatial L_2-quantile is the mean vector. It is very surprising, however, that the robustness properties of spatial quantiles have barely been considered in the literature. To the best of our knowledge, the breakdown point (BDP) has been investigated for the spatial outlyingness measure above ([10], [23], [32]), but not for the spatial quantiles themselves (since the outlyingness measure actually has an explicit expression, obtaining the BDP of this measure is significantly easier than for spatial quantiles). More precisely, the only spatial quantile for which the BDP is known is the spatial median itself; see [22].

The goal of the present work is therefore to study the breakdown properties of spatial quantiles and some of their extensions. We offer three approaches that show diverse distinctive advantages. In Section 2, we first obtain a lower bound by extending the corresponding spatial median result from [22]; to obtain a matching upper bound, we then adopt a constructive approach, that is conceptually simple and allows us to derive the finite-sample breakdown point of spatial quantiles. In Section 3, we adopt an alternative approach, that provides an upper bound on the BDP under contamination in any fixed direction, and that also allows us to determine the breakdown point of L_p-spatial quantiles for any $p > 1$. Last but not least, in Section 4, we use a third approach, that characterizes precisely when breakdown occurs under any given contamination scheme, hence provides the BDPs associated with very diverse contamination scenarios. Quite nicely, this last approach further covers cases where the contamination and/or the ground probability measures are continuous distributions. In Section 5, we show that, in high dimensions, the "practical" BDP will exceed the theoretical one. In Section 6, we briefly summarize the results and provide some final comments. Finally, we collect the proofs in several technical appendices.

2. The BDP of spatial quantiles: a constructive approach

For an order $\alpha \in [0, 1)$, a direction $u \in \mathcal{S}^{d-1}$, and a d-variate sample $x_1, \ldots, x_n$, the breakdown point of the spatial quantile $\mu_{\alpha,u}(x_1, \ldots, x_n)$ is defined as

$$\begin{aligned}&\mathrm{BDP}(\mu_{\alpha,u}; x_1, \ldots, x_n)\\&\quad = \min_{\ell \in \{1,\ldots,n\}} \left\{ \frac{\ell}{n} : \sup_{y} \|\mu_{\alpha,u}(x_1, \ldots, x_n) - \mu_{\alpha,u}(y_1, \ldots, y_n)\| = \infty \right\},\end{aligned}$$

where the supremum is taken over all d-variate samples $y_1, \ldots, y_n$ that differ from $x_1, \ldots, x_n$ by at most ℓ observations (we recall that we use throughout the unique representative of spatial quantiles defined in the introduction). Thus, the BDP is the smallest fraction of the sample one needs to perturb to be able to put the corresponding spatial quantile outside any bounded set of $\mathbb{R}^d$. In [22], it was shown that, for the spatial median (that is, for $\alpha = 0$ and an arbitrary direction u), the BDP is $\lfloor (n+1)/2 \rfloor / n$, which rewrites $\lceil n/2 \rceil / n$.

In this section, our goal is to determine the BDP for any given spatial quantile. We first obtain the following lower bound.

Theorem 2.1. *Fix $\alpha \in [0, 1)$, $u \in \mathcal{S}^{d-1}$ and a positive integer n. Then,*

$$\mathrm{BDP}(\mu_{\alpha,u}; x_1, \ldots, x_n) \geq \left\lceil \frac{n(1-\alpha)}{2} \right\rceil / n$$

for any d-variate sample $x_1, \ldots, x_n$.

This result can be established by adapting the argument in Theorem 2.2 of [22]; see Appendix A for a proof. In the median case, there is no need to derive the corresponding upper bound since any location-equivariant estimator has a BDP that is smaller than or equal to $\lceil n/2 \rceil$ (see Theorem 2.1 of [22]), which, jointly with the lower bound, establishes that the BDP of the spatial median is indeed $\lceil n/2 \rceil$. All spatial quantiles are location-equivariant, so that the upper bound $\lceil n/2 \rceil$ also applies; for $\alpha > 0$, however, this upper bound is not sharp, so that we need some efforts to derive a suitable upper bound.

We will actually show that the lower bound in Theorem 2.1 is sharp, that is, we will show that we can put $\mu_{\alpha,u}(x_1, \ldots, x_n)$ at infinity by perturbing

$$\ell = \left\lceil \frac{n(1-\alpha)}{2} \right\rceil$$

observations. We first provide the following weaker result.

Theorem 2.2. *Fix $\alpha \in [0, 1)$, $u \in \mathcal{S}^{d-1}$ and a positive integer n, and let*

$$\ell := \left\lfloor \frac{n(1-\alpha)}{2} \right\rfloor + 1. \tag{2.4}$$

Then, for any d*-variate sample* $x_1, \ldots, x_n$*, the contaminated sample* $y_1, \ldots, y_n$ *defined by*

$$y_i := \begin{cases} ru & \text{if } i = 1, \ldots, \ell \\ x_i & \text{otherwise} \end{cases} \tag{2.5}$$

provides

$$\mu_{\alpha,u}(y_1, \ldots, y_n) = ru$$

for r *large enough.*

Denoting as $M^y_{\alpha,u}(\mu) = M_{\alpha,u}(\mu; y_1, \ldots, y_n)$ the objective function defining spatial quantiles of the contaminated sample in (2.5) and as $(\partial M^y_{\alpha,u}/\partial v)(\mu)$ the directional derivative of this function at μ in direction v, the proof proceeds by showing that

$$\liminf_{r\to\infty} \min_{v \in \mathcal{S}^{d-1}} \frac{\partial M^y_{\alpha,u}}{\partial v}(ru) > 0,$$

which, from convexity, implies that $\mu_{\alpha,r}(y_1, \ldots, y_n) = ru$ is the unique minimizer of $M^y_{\alpha,u}$ for r large enough; see Appendix A for details. Of course, in view of the lower bound in Theorem 2.1, considering arbitrary large values of r in Theorem 2.2 readily proves the following result.

Corollary 2.1. *Fix* $\alpha \in [0, 1)$*,* $u \in \mathcal{S}^{d-1}$ *and a positive integer* n*. Then, we have the following: (i) if* $n(1-\alpha)/2$ *is not an integer, then*

$$\text{BDP}(\mu_{\alpha,u}; x_1, \ldots, x_n) = \left\lceil \frac{n(1-\alpha)}{2} \right\rceil / n$$

for any d*-variate sample* $x_1, \ldots, x_n$*; (ii) if* $n(1-\alpha)/2$ *is an integer, then*

$$\text{BDP}(\mu_{\alpha,u}; x_1, \ldots, x_n) \in \left\{ \left\lceil \frac{n(1-\alpha)}{2} \right\rceil / n, \left\lceil \frac{n(1-\alpha)}{2} + 1 \right\rceil / n \right\}$$

for any d*-variate sample* $x_1, \ldots, x_n$*.*

While this result is sufficient to determine the asymptotic BDP of spatial quantiles, which is given by

$$\lim_{n\to\infty} \text{BDP}(\mu_{\alpha,u}; x_1, \ldots, x_n) = \frac{1-\alpha}{2},$$

the finite-sample BDP remains unclear in the framework of Corollary 2.1(ii). To improve on this, we present the following result, whose proof is much more involved than that of Theorem 2.2.

Theorem 2.3. *Fix* $\alpha \in [0, 1)$*,* $u \in \mathcal{S}^{d-1}$ *and a positive integer* n *such that* $n(1-\alpha)/2$ *is an integer, and let*

$$\ell := \left\lceil \frac{n(1-\alpha)}{2} \right\rceil.$$

Then, (i) for any d*-variate sample* $x_1, \ldots, x_n$ *such that not all* $x_{\ell+1}, \ldots, x_n$ *belong to the straight line* $\mathcal{L} := \{m + \lambda u : \lambda \in \mathbb{R}\}$*, with* $m := \frac{1}{n-\ell}\sum_{i=\ell+1}^n x_i$*, the contaminated sample* $y_1, \ldots, y_n$ *defined by*

$$y_i := \begin{cases} m + ru & \text{if } i = 1, \ldots, \ell \\ x_i & \text{otherwise} \end{cases} \tag{2.6}$$

provides

$$\mu_{\alpha,u}(y_1, \ldots, y_n) = m + ru$$

for r *large enough; (ii) for any* d*-variate sample* $x_1, \ldots, x_n$ *such that* $x_{\ell+1}, \ldots, x_n$ *belong to* $\mathcal{L}$*, the contaminated sample* $y_1, \ldots, y_n$ *in (2.6) provides*

$$\mu_{\alpha,u}(y_1, \ldots, y_n) = m + \frac{\max(u'(x_{\ell+1} - m), \ldots, u'(x_n - m)) + r}{2} u$$

for r large enough.

For the value of ℓ considered in this result, some samples $x_1, \ldots, x_n$ are such that the corresponding contaminated sample in (2.5) will yield

$$\liminf_{r\to\infty} \min_{v\in\mathcal{S}^{d-1}} \frac{\partial M^y_{\alpha,u}}{\partial v}(ru) < 0,$$

so that $\mu_{\alpha,u}(y_1, \ldots, y_n) \neq ru$. This makes it necessary to consider the contaminated sample in (2.6). The reason why the proof of Theorem 2.3 is more complicated than the one of the previous result is that, for the contaminated sample in (2.6),

$$\liminf_{r\to\infty} \min_{v\in\mathcal{S}^{d-1}} \frac{\partial M^y_{\alpha,u}}{\partial v}(m+ru) = 0,$$

which does not allow one to conclude as above. Instead, the proof requires a second-order expansion of the minimal directional derivative to show that

$$\min_{v\in\mathcal{S}^{d-1}} \frac{\partial M^y_{\alpha,u}}{\partial v}(m+ru) = \frac{1}{2nr^2}\sum_{i=\ell+1}^{n}\left(\|x_i - m\|^2 - (u'(x_i - m))^2\right) + O\left(\frac{1}{r^3}\right).$$

In the framework of Theorem 2.3(i), this guarantees that this minimal directional derivative is positive for r large enough, which, from convexity, establishes the result (Theorem 2.3(ii) rather follows by directly computing the quantile in the specific case considered there); see Appendix A for details.

More importantly, Theorem 2.3 allows us to refine Corollary 2.1 into the following result, which fully settles the question of the finite-sample breakdown point of spatial quantiles.

Corollary 2.2. *Fix $\alpha \in [0, 1)$, $u \in \mathcal{S}^{d-1}$ and a positive integer n. Then,*

$$\mathrm{BDP}(\mu_{\alpha,u}; x_1, \ldots, x_n) = \left\lceil \frac{n(1-\alpha)}{2} \right\rceil / n$$

for any d-variate sample $x_1, \ldots, x_n$.

We conclude this section by illustrating empirically the theorems above; see Figure 1. For $n = 40$, we generated $x_1, \ldots, x_n$ independently from the bivariate normal distribution with mean vector $(1, 1)$ and identity covariance matrix. For different values of ℓ, we then evaluated the spatial quantiles of order $\alpha = .49$ (left panels) and order $\alpha = .5$, twice in direction $u = (1, 0)$, for contaminated samples $y_1, \ldots, y_n$ as in (2.5) (top panels) or as in (2.6) (bottom panels), in each case for increasing values of r; note that only $\alpha = .5$ makes $n(1-\alpha)/2$ an integer. First note that, in all panels, the quantiles associated with $\ell = \lceil n(1-\alpha)/2\rceil$ do not break down, which is compatible with Theorem 2.1. The result in Theorem 2.2 is confirmed by the quantiles associated with $\ell = \lceil n(1-\alpha)/2\rceil$ in the top left panel and those associated with $\ell = \lceil n(1-\alpha)/2\rceil + 1$ in the top right one (these quantiles, for large r, are equal to ru in both panels). It is seen that, in the integer case, the quantiles associated with $\ell = \lceil n(1-\alpha)/2\rceil$ are not equal to ru. As for Theorem 2.3, it is illustrated by the quantiles associated with $\ell = \lceil n(1-\alpha)/2\rceil$ in both bottom panels. Figure 1 therefore fully supports our theoretical results.

Quite nicely, carefully inspecting the proofs of the results from this section reveals that these results also hold in a general, possibly infinite-dimensional, Hilbert space (this will actually be the case for all results of Section 3, too).

3. A directional investigation and an extension to L_p-quantiles

Since lower and upper bounds are matching in Section 2, the approach adopted there considers the worst-case scenario in terms of contamination schemes. Yet it is interesting to evaluate the impact of specific types of contaminations; we refer to Section 5 for a motivation in a high-dimensional context. Also, the approach from Section 2 does not generalize to spatial L_p-quantiles as it requires guessing the value of the corresponding quantile, which is possible only for the standard spatial quantiles obtained with $p = 1$. In this section, we therefore adopt an alternative approach. We have the following result.

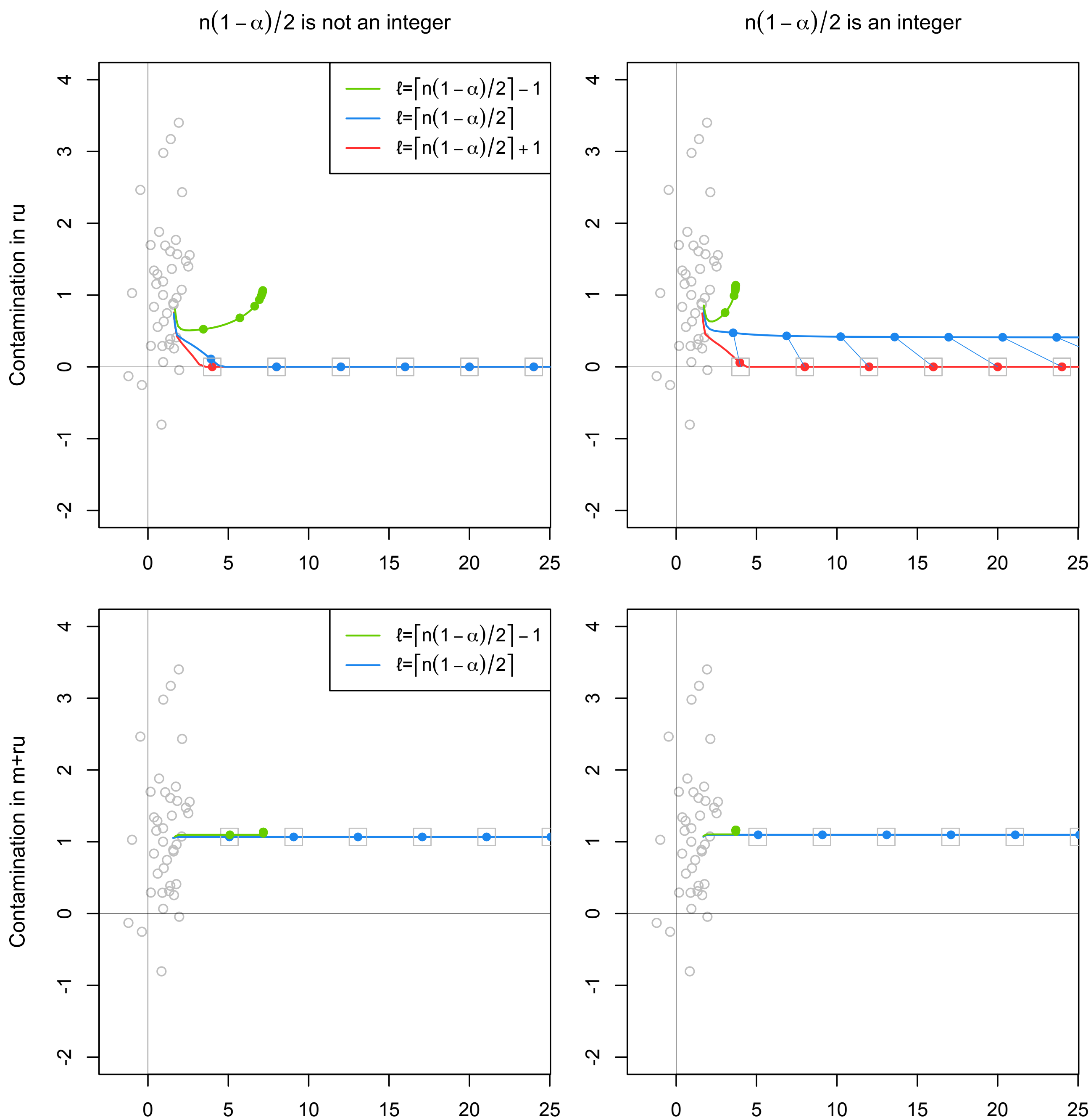

FIG 1. (Top left panel:) Trajectories of $r \mapsto \mu_{\alpha,u}(y_1,\ldots,y_n)$, for $n = 40$, $\alpha = .49$ and $u = (1,0)$, for contaminated samples $y_1,\ldots,y_n$ as in (2.5) for three different contamination levels ℓ (most of the red trajectory is hidden behind the blue one); the uncontamined sample $x_1,\ldots,x_n$ is shown with grey circles. At a few selected values of r, squares show the corresponding values of ru, which are linked through a coloured line segment to the corresponding quantile $\mu_{\alpha,u}(y_1,\ldots,y_n)$. (Top right panel:) the corresponding results for the value $\alpha = .5$, for which $n(1-\alpha)/2$ is an integer. (Bottom left and right panels:) The corresponding results for only two contamination levels ℓ and for contaminated samples $y_1,\ldots,y_n$ as in (2.6), so that the squares show a few selected values of $m + ru$; see Section 2 for details.

Theorem 3.1. *Fix $\alpha \in [0,1)$, $u, v \in \mathcal{S}^{d-1}$ and a positive integer n, and let*

$$\ell := \left\lfloor \frac{n(1-\alpha u'v)}{2} \right\rfloor + 1.$$

Then, for any d-variate sample $x_1, \ldots, x_n$, the quantile $\mu_{\alpha,u}(y_1, \ldots, y_n)$ associated with the contaminated sample defined by

$$y_i := \begin{cases} rv & \text{if } i = 1, \ldots, \ell \\ x_i & \text{otherwise} \end{cases} \tag{3.7}$$

is such that $\|\mu_{\alpha,u}(y_1, \ldots, y_n)\| \to \infty$ as $r \to \infty$.

The proof proceeds by showing that, for a slightly modified objective function $\tilde{M}_{\alpha,u}(\mu; y_1, \ldots, y_n)$ admitting the same minimizers as the objective function $M_{\alpha,u}(\mu; y_1, \ldots, y_n)$ defining the spatial quantile $\mu_{\alpha,u}(y_1, \ldots, y_n)$, we have

$$\tilde{M}_{\alpha,u}(rv; y_1, \ldots, y_n) < \inf_{\mu: \|\mu\| < \sqrt{r}} \tilde{M}_{\alpha,u}(\mu; y_1, \ldots, y_n),$$

which of course guarantees that, for r large enough, $\|\mu_{\alpha,u}(y_1, \ldots, y_n)\| \geq \sqrt{r}$; see Appendix B. Now, in view of the lower bound from Theorem 2.1, taking $v = u$ in Theorem 3.1 provides an alternative proof of Corollary 2.1; this strategy of proof, however, does not allow one to conclude about the BDP when $n(1-\alpha)/2$ is an integer (that is, by proceeding as in the proof of Theorem 3.1, one cannot strengthen Corollary 2.1 into Corollary 2.2). Theorem 3.1 is still interesting for both following reasons.

First, Theorem 3.1 provides an upper bound on the BDP depending on the direction v in which contamination occurs, as it states that, when contamination occurs in direction v, it is sufficient to perturb

$$\left\lfloor \frac{n(1-\alpha u'v)}{2} \right\rfloor + 1 \tag{3.8}$$

observations to break the quantile. Quite intuitively, this is a monotone non-decreasing function of $u'v$: the more the contamination occurs in a direction v that is opposed to u (that is, the smaller $u'v$), the larger the amount of contamination needed to break the quantile in Theorem 3.1. A natural question, that will be tackled in the next section, is whether the directional upper bound (3.8) is sharp. Note that this upper bound is sharp at least when $v = u$ and $n(1-\alpha u'v)/2$ is not an integer, since it then coincides with the lower bound and upper bound from Theorem 2.1 and Theorem 2.2, respectively.

Second, the strategy of proof adopted in Theorem 3.1 allows us to get a strong result on the BDP of spatial L_p-quantiles for $p > 1$ (as mentioned above, the constructive approach from Section 2 does not allow one to obtain results on the robustness of L_p-quantiles). We have the following result (see Appendix B for a proof inspired by the one of the previous theorem).

Theorem 3.2. *Fix $\alpha \in [0,1)$, $u, v \in \mathcal{S}^{d-1}$ and a positive integer n. Then, for any $p > 1$ and d-variate sample $x_1, \ldots, x_n$, the L_p-quantile $\mu_{\alpha,u,p}(y_1, \ldots, y_n)$ associated with the contaminated sample defined by*

$$y_i := \begin{cases} rv & \text{if } i = 1 \\ x_i & \text{otherwise} \end{cases} \tag{3.9}$$

is such that $\|\mu_{\alpha,u,p}(y_1, \ldots, y_n)\| \to \infty$ as $r \to \infty$.

Quite remarkably, irrespective of the direction v in which contamination occurs, it is thus sufficient to perturb a single observation to break spatial L_p-quantiles with $p > 1$. This is illustrated numerically in Figure 2. For the same sample $x_1, \ldots, x_n$ as in Figure 1, we report there, for $p \in \{1.5, 2, 3, 4\}$, the trajectories of $r \mapsto \mu_{\alpha,u,p}(y_1, \ldots, y_n)$, with $\alpha = .5$ and $u = (1,0)$, computed from contaminated samples $y_1, \ldots, y_n$ as in (3.9) with $v = u = (1,0)$, $v = (0,1)$, and $v = -u = (-1,0)$. As expected from Theorem 3.2, breakdown occurs for any combination of the values of p and v considered.

Of course, Theorem 3.2 directly provides the following result.

Corollary 3.1. *Fix $\alpha \in [0,1)$, $u \in \mathcal{S}^{d-1}$ and a positive integer n. Then,*

$$\mathrm{BDP}(\mu^p_{\alpha,u}; x_1, \ldots, x_n) = \frac{1}{n}$$

for any $p > 1$ and any d-variate sample $x_1, \ldots, x_n$.

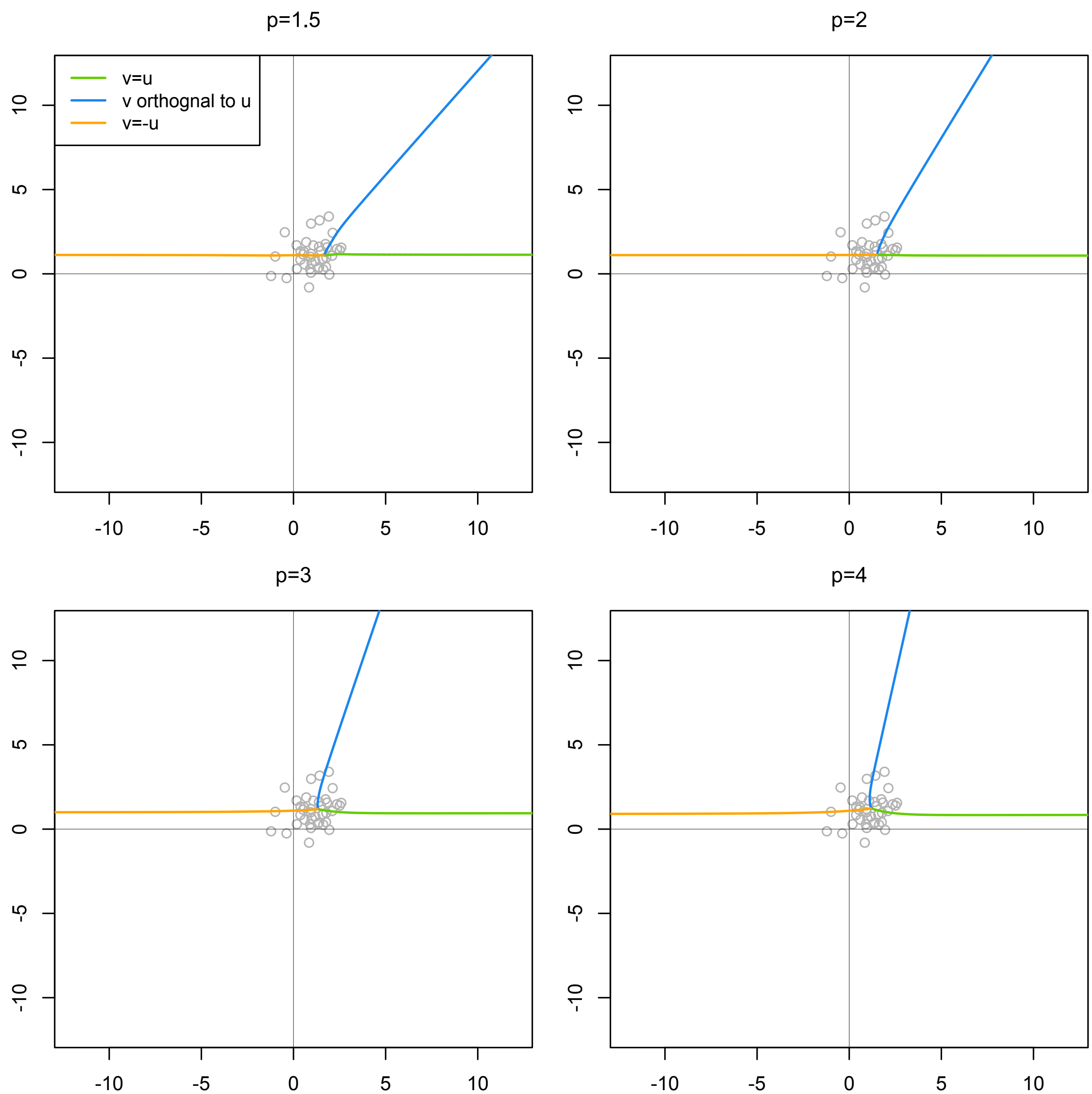

FIG 2. Trajectories of $r \mapsto \mu_{\alpha,u,p}(y_1,\ldots,y_n)$ for $p \in \{1.5, 2, 3, 4\}$, with $\alpha = .5$ and $u = (1,0)$, computed from contaminated samples $y_1,\ldots,y_n$ as in (3.9) with $v = u = (1,0)$, $v = (0,1)$, and $v = -u = (-1,0)$; the original sample $x_1,\ldots,x_n$ is the same as in Figure 1.

Clearly, it follows that, from a robustness point of view, spatial quantiles are to be strongly favored over their L_p-counterparts with $p > 1$.

In Theorems 3.1–3.2 above, contamination occurs in a single direction v. We conclude this section by showing that the poor robustness of L_p-quantiles also materializes when placing contamination symmetrically in directions v and $-v$ (the case of spatial quantiles, that is much more complicated, will be considered in Section 4 below). We have the following result (see Appendix B for a proof).

Theorem 3.3. *Fix $\alpha \in (0,1)$, $u, v \in \mathcal{S}^{d-1}$ and a positive integer n. Then, for any $p > 1$ and d-variate sample $x_1, \ldots, x_n$, the L_p-quantile, $\mu_{\alpha,u,p}(P_n)$ say, associated with the contaminated probability measure defined by*

$$P_n := \frac{1}{2n}\delta_{rv} + \frac{1}{2n}\delta_{-rv} + \frac{1}{n}\sum_{i=2}^{n}\delta_{x_i} \tag{3.10}$$

is such that $\|\mu_{\alpha,u,p}(P_n)\| \to \infty$ as $r \to \infty$ (here, δ_a denotes the Dirac probability measure at a).

In line with Theorem 3.2, the result thus shows that a fraction $1/n$ of (here, symmetric) contamination is enough to break L_p-quantiles[1] when $p > 1$. We provide a numerical illustration in Figure 3. Still for the same sample $x_1, \ldots, x_n$ as in Figure 1, we plot there, for $p \in \{1.5, 2, 3, 4\}$, the trajectories of $r \mapsto \mu_{\alpha,u,p}(P_n)$ for $p \in \{1.5, 2, 3, 4\}$, with $\alpha = .5$ and $u = (1,0)$, computed from contaminated probability measures P_n as in (3.10) with $v = u = (1,0)$, $v = (1/\sqrt{2}, 1/\sqrt{2})$, and $v = (0,1)$. Clearly, irrespective of the direction v of the symmetric contamination considered, all L_p-quantiles break, as predicted in Theorem 3.3.

4. A sharp approach

The previous section considered robustness of spatial quantiles when contamination occurs in a fixed direction v and derived an upper bound on the corresponding BDP. This of course leaves some space for improvement. In particular, we may wonder about the exact value of such a directional BDP. Also, we may consider robustness of spatial quantiles under more general contamination schemes, e.g. under symmetric contamination schemes that, for a unit vector v and for r large, would put half of the contamination in rv and the other half in $-rv$ (in line, thus, with the contamination pattern considered in Theorem 3.3). In this section, we provide a general result that *characterizes* when breakdown occurs when the ground probability measure (the one associated with the original sample $x_1, \ldots, x_n$) is contaminated in an essentially arbitrary way, through a contamination measure. The result even allows the ground probability and/or contamination measures to be absolutely continuous with respect to the Lebesgue measure (see Appendix C for a proof).

Theorem 4.1. *Fix sequences (Q_k) and (Λ_k) of probability measures on $\mathbb{R}^d$. Assume that (Q_k) converges weakly to a probability measure Q and that $\Lambda_k(K) \to 0$ for any compact subset K of $\mathbb{R}^d$ as $k \to \infty$ with*

$$w_k := \int_{\mathbb{R}^d \setminus \{0\}} \frac{z}{\|z\|}\, d\Lambda_k(z) \to w$$

for some $w \in \mathbb{R}^d$ as $k \to \infty$. Fix $c \in [0,1)$ and define $P_k := (1-c)Q_k + c\Lambda_k$ for all k. Fix $\alpha \in [0,1)$ and $u \in \mathcal{S}^{d-1}$, and define $\tilde{\alpha} = \|\alpha u + cw\|/(1-c)$. Let $\tilde{u} = (\alpha u + cw)/\|\alpha u + cw\|$ if $\tilde{\alpha} > 0$ and $\tilde{u}$ be arbitrary in $\mathcal{S}^{d-1}$ if $\tilde{\alpha} = 0$. Then, the following holds.

(i) If $\tilde{\alpha} > 1$, then $\|\mu_{\alpha,u}(P_k)\| \to \infty$ as $k \to \infty$.
(ii) If $\tilde{\alpha} = 1$, then $\|\mu_{\alpha,u}(P_k)\| \to \infty$ as $k \to \infty$, provided either a) Q is not supported on a halfline with direction $-\tilde{u}$, or b) there exists a bounded sequence (z_k) such that, for all k, Q_k is supported on the halfline $\{z_k - \lambda\tilde{u} : \lambda \geq 0\}$ and Λ_k is supported on the halfline $\{z_k + \lambda\tilde{u} : \lambda \geq r_k\}$ with $r_k \to \infty$.
(iii) If $\tilde{\alpha} < 1$, then the sequence $(\mu_{\alpha,u}(P_k))$ is bounded. In addition, if Q is not supported on a line with direction $\tilde{u}$, then $\mu_{\alpha,u}(P_k) \to \mu_{\tilde{\alpha},\tilde{u}}(Q)$ as $k \to \infty$.

When applying this result to determine BDPs, Q_k will play the role of the ground probability measure, whereas Λ_k will stand for the contamination measure; in this framework, note that w_k is the mean direction associated with the contamination measure. In particular, $w_k = 0$ for symmetric contamination patterns, and, more generally, $w = 0$ for contamination patterns that are eventually symmetric.

An examination of the proof of Theorem 4.1 shows that Part (i) does not require that (Q_k) converges weakly: it holds for an arbitrary sequence (Q_k) of probability measures. Similarly, the boundedness result in (iii) holds as soon as (Q_k) is tight. It also worth mentioning that only case b) of Part (ii) requires taking $\mu_{\alpha,u}(P_k)$ as the barycenter of the collection of minimizers of the objective function in (1.1); all other statements in Theorem 4.1 are valid for an *arbitrary* sequence $(\mu_{\alpha,u}(P_k))$ of quantiles of order α in direction u for (P_k).

[1] A difference with Theorem 3.2, however, is that the case $\alpha = 0$ is excluded in Theorem 3.3, which is quite natural (such symmetric contaminations are not expected to break the innermost L_p-quantile).

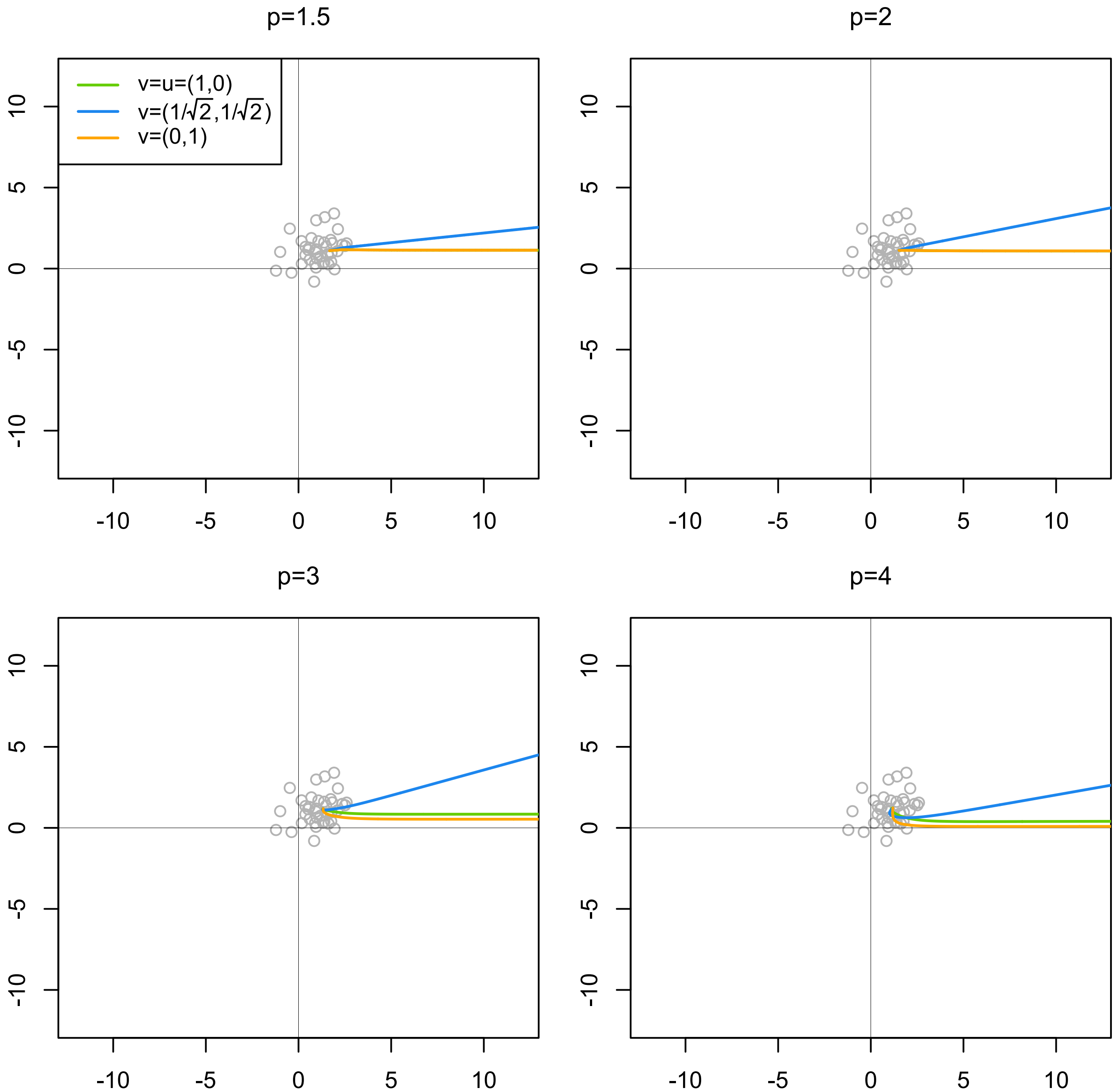

FIG 3. Trajectories of $r \mapsto \mu_{\alpha,u,p}(P_n)$ for $p \in \{1.5, 2, 3, 4\}$, with $\alpha = .5$ and $u = (1,0)$, computed from contaminated probability measures P_n as in (3.10) with $v = u = (1,0)$, $v = (1/\sqrt{2}, 1/\sqrt{2})$, and $v = (0,1)$; the original sample $x_1, \ldots, x_n$ is the same as in Figure 1. In both upper panels, the green trajectory coincides with the orange one.

This result, that characterizes when breakdown occurs, has several important consequences. In particular, as it may be expected from the characteristic nature of the result, Theorem 4.1 yields an alternative proof of Corollary 2.2, as we now briefly explain. Let $Q_k \equiv Q$ be the empirical probability measure of $y_{\ell+1} = x_{\ell+1}, \ldots, y_n = x_n$ and Λ_k be the empirical probability measure of $y_1, \ldots, y_\ell = kv$, with $v \in \mathcal{S}^{d-1}$. With $c = \ell/n$, the resulting probability measure P_k is the empirical probability measure of $y_1, \ldots, y_n$. Assume here, for the sake of simplicity, that $x_{\ell+1}, \ldots, x_n$ are not supported on a single line. Then breakdown will occur if and only if

$$\tilde{\alpha} = \frac{\|\alpha u + (\ell/n) v\|}{1 - (\ell/n)} = \frac{\sqrt{\alpha^2 + (\ell/n)^2 + 2\alpha(\ell/n)u'v}}{1 - (\ell/n)} \geq 1,$$

that is, breakdown can be achieved by perturbing

$$\left\lceil \frac{n(1-\alpha^2)}{2(1+\alpha u'v)} \right\rceil \tag{4.11}$$

observations but not less. The smallest value of (4.11) is obtained for $v = u$, so that breakdown can be achieved by perturbing

$$\ell = \left\lceil \frac{n(1-\alpha^2)}{2(1+\alpha)} \right\rceil = \left\lceil \frac{n(1-\alpha)}{2} \right\rceil$$

observations but not less, in line with Corollary 2.2. Using Theorem 4.1, we provide in Appendix C a detailed proof of Corollary 2.2 that applies to arbitrary contamination schemes which, in particular, do not make any assumption on $x_1, \ldots, x_n$.

If one restricts to contamination in a fixed direction v, then the argument above implies that the corresponding BDP is

$$\left\lceil \frac{n(1-\alpha^2)}{2(1+\alpha u'v)} \right\rceil / n. \tag{4.12}$$

It is interesting to compare this with the result in Theorem 3.1, that shows that an upper bound for the BDP in direction v is

$$\left(\left\lfloor \frac{n(1-\alpha u'v)}{2} \right\rfloor + 1 \right) / n.$$

Focusing, for the sake of simplicity, on asymptotic BDPs, Theorem 3.1 provides an upper bound on the asymptotic BDP in direction v given by

$$\frac{1-\alpha u'v}{2}$$

(see (3.8)), whereas, in view of (4.12), the sharp approach from this section shows that the asymptotic BDP in direction v is actually equal to

$$\frac{(1-\alpha^2)}{2(1+\alpha u'v)} = \frac{1-\alpha u'v}{2} \times \frac{1-\alpha^2}{1-\alpha^2(u'v)^2} \leq \frac{1-\alpha u'v}{2} \cdot$$

Therefore, the asymptotic upper bound from Theorem 3.1 is sharp if and only if $|u'v| = 1$, and the smaller $|u'v|$ is, the less sharp this upper bound is.

Parallel to the upper bound resulting from Theorem 3.1, the BDP in direction v in (4.12) is a monotone non-increasing function of $u'v$, which can interpreted in the same way as in Section 3. The BDPs in direction $v = u$, in an arbitrary direction v orthogonal to u, and in direction $v = -u$ are given by

$$\ell_u := \left\lceil \frac{n(1-\alpha)}{2} \right\rceil / n, \quad \ell_\perp := \left\lceil \frac{n(1-\alpha^2)}{2} \right\rceil / n, \quad \text{and} \quad \ell_{-u} := \left\lceil \frac{n(1+\alpha)}{2} \right\rceil / n, \tag{4.13}$$

respectively. Also, for arbitrary "symmetric" contaminations, i.e. those providing $w = 0$ in Theorem 4.1 (such as, e.g., those distributing half of the contamination in rv and half of it in $-rv$ for a given unit vector v), the same reasoning as above shows that the corresponding BDP is

$$\ell_{\text{symm}} := \lceil n(1-\alpha) \rceil / n; \tag{4.14}$$

see Figure 4. A numerical illustration is provided in Figure 5. We show there trajectories of the form $r \mapsto \mu_{\alpha,u}(y_1, \ldots, y_n)$, still with $\alpha = .5$ and $u = (1, 0)$, for contamined samples $y_1, \ldots, y_n$ obtained by combining the four contamination levels $\ell_u, \ell_\perp, \ell_{-u}$ and ℓ_{symm} with the four types of contaminations described above (that is, the type of contamination in (3.7) for $v = u = (1, 0)$, $v = (0, 1)$, $v = -u = (-1, 0)$, and the symmetric contamination that has $y_1 = \ldots = y_{\ell/2} = r(0, 1)$, $y_{(\ell/2)+1} = \ldots = y_\ell = r(0, -1)$ and $y_i = x_i$ for $i > \ell$). The results are perfectly in line with the various BDPs above: for each type of contamination, indeed, the quantile breaks when the number of contaminated observations is larger than or equal to (n times) the corresponding BDP in (4.13)–(4.14) above.

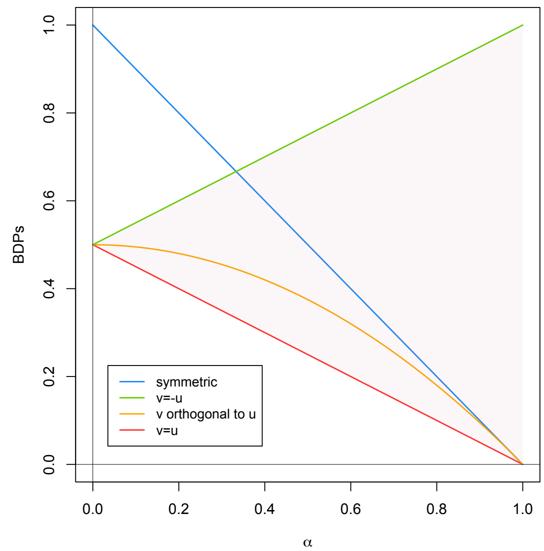

FIG 4. Plots, as functions of α, of the BDP of $\mu_{\alpha,u}$ in any direction u for contaminations in direction $v = u$ (ℓ_u; red), for contaminations in an arbitrary direction orthogonal to u ($\ell_\perp$; blue), for contaminations in direction $v = -u$ (ℓ_{-u}; green), and for an arbitrary symmetric contaminations ($\ell_{\rm symm}$; orange). See (4.13)–(4.14).

5. High dimensions

As we learned in the previous sections, the BDP for the spatial quantile of order α in direction u is $\lfloor n(1-\alpha)/2 \rfloor / n$ when all types of contaminations are considered, but this increases to

$$\left\lceil \frac{n(1-\alpha^2)}{2(1+\alpha u'v)} \right\rceil / n \tag{5.15}$$

when one restricts to contaminations in direction v. While this may seem to be of academic interest only, it is actually practically relevant in high dimensions: in high-dimensional scenarios indeed, the probability mass (hence also, the possible contamination) will increasingly concentrate on the orthogonal complement to any given direction (see, e.g., [14] and the discussion therein), and in particular to the direction u in which the quantile is considered. This will result into a practical BDP given by

$$\left\lceil \frac{n(1-\alpha^2)}{2} \right\rceil / n \tag{5.16}$$

(which is larger than $\lfloor n(1-\alpha)/2 \rfloor / n$). One might argue, however, that while the contamination will eventually take place in the orthogonal complement to u (denote it as $\{u\}^\perp$), there is no reason why contamination should concentrate along a single direction $v \in \{u\}^\perp$, so that it might be misleading to consider that the practical BDP in high dimensions is (5.16). This is an important motivation to determine the BDP associated with an *arbitrary* contamination that eventually concentrates on $\{u\}^\perp$.

To do so, consider thus, in the framework of Theorem 4.1, a sequence of contamination measures (Λ_k) that eventually concentrates on $\{u\}^\perp$, so that the corresponding sequence (w_k) is such that $u'w_k \to 0$ as $k \to \infty$. Assuming that the sequence (w_k) converges in $\mathbb{R}^d$, its limit w thus belongs to the unit ball in $\{u\}^\perp$. The special case for which $w = v \in \mathcal{S}^{d-1} \cap \{u\}^\perp$ provides the case where the contamination is eventually along a single direction v of $\{u\}^\perp$, which yields the BDP in (5.16). Assume thus that $\|w\| < 1$. It then follows from Theorem 4.1 that the spatial quantile $\mu_{\alpha,u}$ will break down if and only if

$$\tilde{\alpha} = \frac{\|\alpha u + (\ell/n) w\|}{1-(\ell/n)} = \frac{\sqrt{\alpha^2 + (\ell/n)^2 \|w\|^2}}{1-(\ell/n)} \geq 1.$$

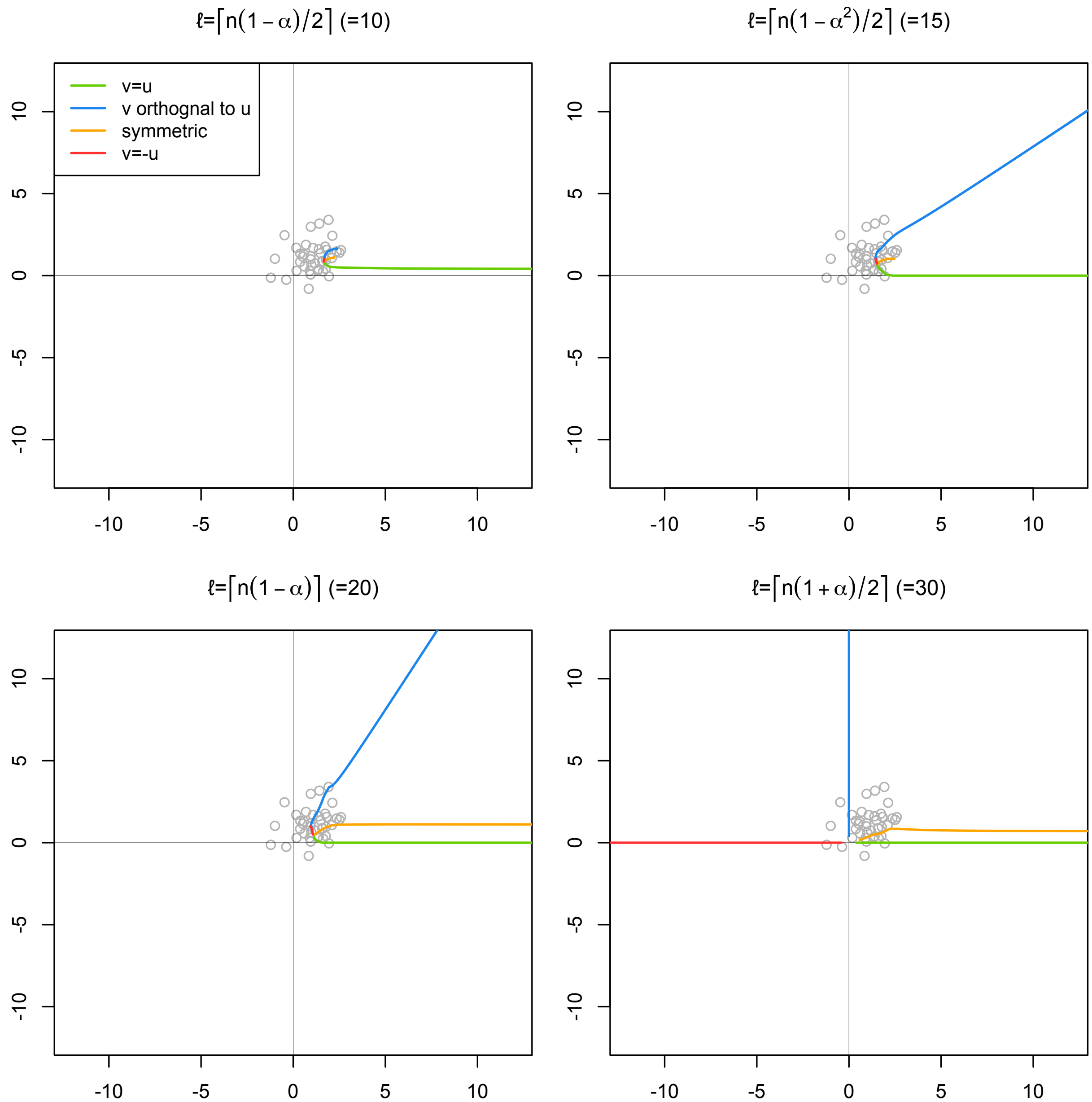

FIG 5. Trajectories of $r \mapsto \mu_{\alpha,u}(y_1,\ldots,y_n)$, with $\alpha = .5$ and $u = (1,0)$, computed from contaminated samples $y_1,\ldots,y_n$ as in (3.9) with $v = u = (1,0)$, $v = (0,1)$, and $v = -u = (-1,0)$, and from samples $y_1,\ldots,y_n$ that are contaminated symmetrically in directions $(0,1)$ and $(0,-1)$. The various panels differ by the number ℓ of contaminated observations. In each case, the original sample $x_1,\ldots,x_n$ is the same as in Figure 1; see Section 4 for details.

Since it is easy to check that this condition rewrites

$$\ell \geq \left\lceil \frac{-2n + \sqrt{4n^2 - 4n^2(1-\alpha^2)(1-\|w\|^2)}}{-2(1-\|w\|^2)} \right\rceil,$$

the BDP in the considered high-dimensional scenario is

$$\left\lceil \frac{(1-\alpha^2)n}{1+\sqrt{1-(1-\alpha^2)(1-\|w\|^2)}} \right\rceil / n. \tag{5.17}$$

Interestingly, this is a monotone decreasing function of $\|w\|$, so that the BDP in (5.16), that is associated with contamination in a given direction $v \in \{u\}^\perp$, actually corresponds to the worst-case scenario. The BDP in high dimensions can be as large as $\lceil n(1-\alpha)\rceil/n$, which is the value of (5.17) obtained for $w=0$. Remarkably, this (very large!) maximal value of the BDP is obtained when the contamination in $\{u\}^\perp$ is symmetric, which, in view of the results in [14], is the rule rather than the exception.

To illustrate numerically the fact that the BDP is practically larger than expected in high dimensions, we conducted the following simulation exercise. We considered spatial quantiles of order $\alpha = .5$ in direction $u = (1, 0, \ldots, 0) \in \mathbb{R}^d$. For $n = 200$ and various dimensions d, we generated independently observations $x_1, \ldots, x_n$ from the d-variate standard normal distribution and a direction v from the uniform distribution over $\mathcal{S}^{d-1}$. Denoting as $\mu_0(x_1, \ldots, x_n)$ the spatial median of $x_1, \ldots, x_n$, we then computed

$$\ell_* := \min\left\{\ell : \frac{\|\mu_{\alpha,u}(Rv, \ldots, Rv, x_{\ell+1}, \ldots, x_n) - \mu_0(x_1, \ldots, x_n)\|}{\|\mu_{\alpha,u}(x_1, \ldots, x_n) - \mu_0(x_1, \ldots, x_n)\|} > C\right\} \tag{5.18}$$

(we used $R = 10^6$ and $C = 1000$), which is an empirical estimate of the BDP when the sample $x_1, \ldots, x_n$ is contaminated in direction v. In each dimension $d \in \{3, 10, 40, 200\}$, we repeated this $4{,}000$ times and plotted the histogram of the resulting $4{,}000$ values of ℓ_*; see Figure 6. In each dimension, we also plotted the density of

$$\frac{(1-\alpha^2)}{2(1+\alpha u'V)}$$

in direction V, where V is uniform over $\mathcal{S}^{d-1}$ (conditional on V, this is the asymptotic BDP under contamination in direction V); this density was obtained by using the fact that $u'V$ has a symmetric distribution such that $(u'V)^2 \sim \mathrm{Beta}(1/2, (d-1)/2)$ (see, e.g., [1], page 54). Obviously, the excellent fit confirms our theoretical results, as well as the practical strategy in (5.18) to estimate the BDP (in this respect, empirical results are robust to the specific choices of R and C). For any dimension d, the histogram expands from the minimal BDP value $\lceil n(1-\alpha)/2\rceil/n$ to the maximal BDP value $\lceil n(1+\alpha)/2\rceil/n$ that are associated with contamination in a single direction v, that is the lower and upper bounds in v of the BDP in (5.15). As the dimension d increases, however, the histogram concentrates around the BDP value $\lceil n(1-\alpha^2)/2\rceil/n$, as expected, so that this value may be considered the practical BDP in high dimensions. Importantly, recall that the present simulation exercise focuses on the worst-case scenario where contamination in high dimensions would concentrate in a single direction of $\{u\}^\perp$: as we showed above, any other contamination schemes that asymptotically concentrate on $\{u\}^\perp$ will actually provide larger practical BDPs.

6. Wrap up and final comments

Being a concept of an L_1-nature, spatial (or geometric) quantiles naturally enjoy some robustness to possible outlying observations. In view of the important success met by these quantiles in the last decades, it is most surprising that their robustness properties have not been investigated in the literature. This provided a motivation to study in this paper the breakdown properties of spatial quantiles in any dimension d.

Spatial quantiles reduce to the traditional univariate quantiles for $d = 1$, and our results reveal that part of their robustness properties are indeed inherited from their classical univariate antecedents: spatial quantiles of order α in direction u have a BDP equal to

$$\left\lceil \frac{n(1-\alpha)}{2} \right\rceil / n$$

when all types of contaminations are allowed for, and a BDP equal to

$$\left\lceil \frac{n(1-\alpha)}{2} \right\rceil / n \quad \text{and} \quad \left\lceil \frac{n(1+\alpha)}{2} \right\rceil / n \tag{6.19}$$

when restricting to contaminations in direction u and $-u$, respectively (in the univariate case, contamination can be considered only in the direction u in which the quantile is computed or in the opposite direction $-u$); comparison with the BDPs of univariate quantiles of course requires adopting for these the center-outward parametrization used for spatial quantiles.

Of course, spatial quantiles were designed to tackle the multivariate case, where, interestingly, much richer types of contaminations may be considered. Our results imply that, when focusing on contaminations with mean direction w (in

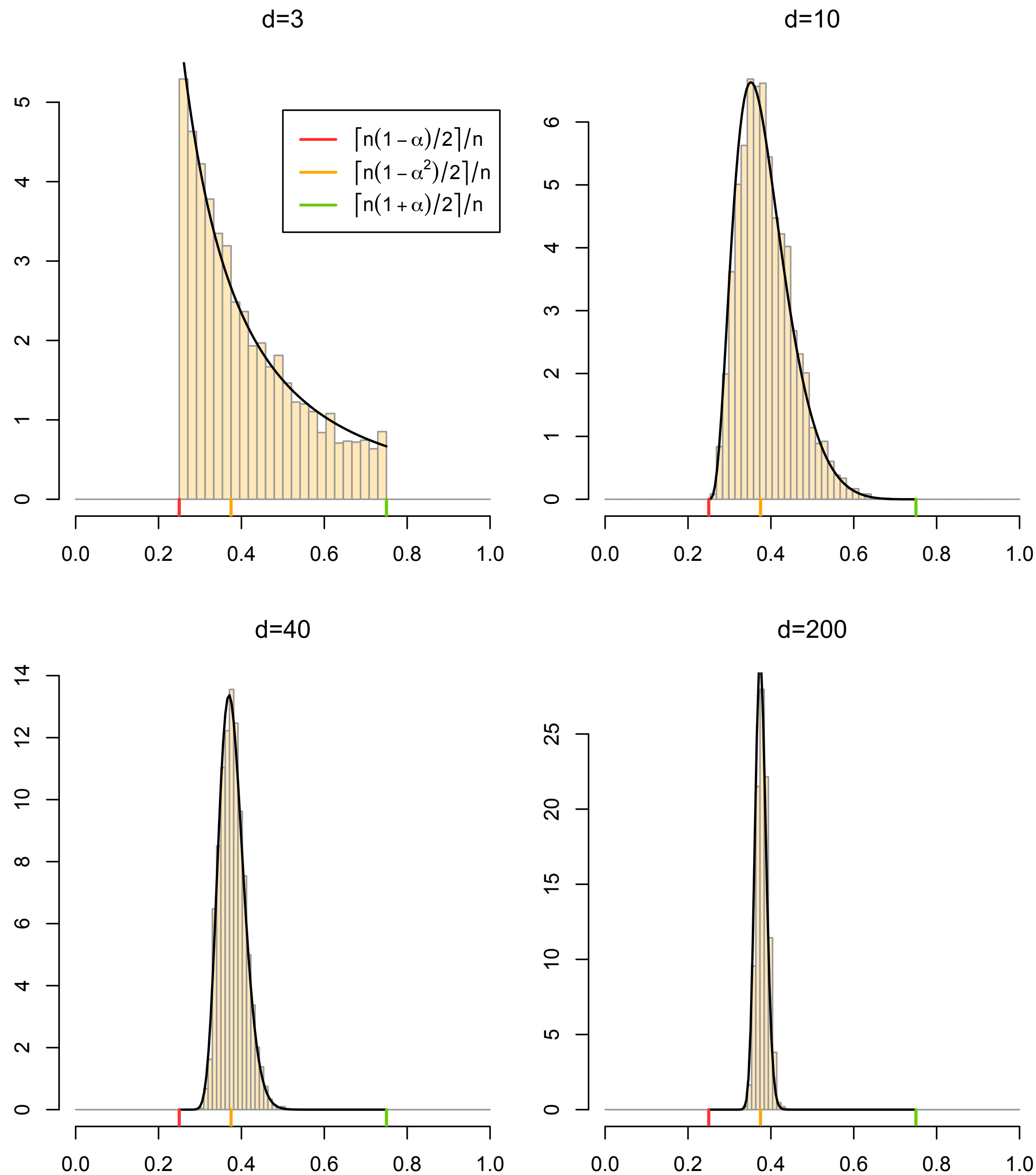

FIG 6. (Top left panel:) For $d = 3$, this provides the histogram of the BDP estimates in (5.18) (from $4,000$ replications) when evaluating the spatial quantile of order $\alpha = .5$ in direction $u = (1, 0, \ldots, 0) \in \mathbb{R}^d$ for standard normal samples of size $n = 200$ contaminated in a direction v that is randomly sampled from the uniform distribution over $\mathcal{S}^{d-1}$. The density described in Section 5 is plotted in black, and three important values of the BDP are marked in red, blue and green. (Other panels): the corresponding results for dimensions $d = 10, 40, 200$.

the sense of Theorem 4.1), the BDP of spatial quantiles of order α in direction u is

$$\left\lceil \frac{n(1-\alpha^2)}{(1+\alpha u'w) + \sqrt{(1+\alpha u'w)^2 - (1-\alpha^2)(1-\|w\|^2)}} \right\rceil / n$$

(this general BDP, that has not been derived in the earlier sections, can be obtained from Theorem 4.1). Taking $w = v \in \mathcal{S}^{d-1}$ provides contaminations in a given direction v, which yields the BDP

$$\left\lceil \frac{n(1-\alpha^2)}{2(1+\alpha u'v)} \right\rceil / n,$$

that is achieving a continuum of values between both "univariate" extreme ones in (6.19). Considering rather $u'w = 0$ provides contaminations in the orthogonal complement to the direction u in which the spatial quantile is considered, which, as explained in Section 5, is the natural case in high dimensions; the resulting BDP, namely

$$\left\lceil \frac{n(1-\alpha^2)}{1+\sqrt{1-(1-\alpha^2)(1-\|w\|^2)}} \right\rceil / n \tag{6.20}$$

then ranges between $\lceil n(1-\alpha^2)/2 \rceil / n$ and $\lceil n(1-\alpha) \rceil / n$. This reveals that spatial quantiles may show a remarkably high robustness, particularly in high dimensions, where contaminations will tend to be symmetric in the orthogonal complement to u ([14]), a framework that provides the maximal value $\lceil n(1-\alpha) \rceil / n$ of the BDP in (6.20).

While these results provide a thorough investigation of the breakdown properties of spatial quantiles, there remain two open questions that could be considered in future research work: (1) when breakdown occurs, it would be interesting to characterize in which directions spatial quantiles go to infinity. While spatial quantiles probably go to infinity in direction v when contamination is in direction v indeed, it is much less easy to guess what happens for symmetric contaminations schemes. (2) Remember that the results of Sections 2–3 hold in general, possibly infinite-dimensional, Hilbert spaces (to the best of our knowledge, these actually provide the first robustness results on infinite-dimensional quantiles). Another avenue for future research would then be to extend Theorem 4.1 to general Hilbert spaces, too. The technical challenges this raises, that are associated with the lack of compactness of the unit sphere in infinite-dimensional Hilbert spaces, did not allow us to consider this extension in the framework of the present paper.

Appendix A: Proofs for Section 2

In this first appendix, we prove Theorems 2.1–2.3.

PROOF OF THEOREM 2.1. Fix an arbitrary d-variate sample $x_1, \ldots, x_n$. Since $\mu_{\alpha,u}$ is translation-equivariant,

$$\text{BDP}(\mu_{\alpha,u}; x_1 + t, \ldots, x_n + t) = \text{BDP}(\mu_{\alpha,u}; x_1, \ldots, x_n)$$

for any $t \in \mathbb{R}^d$ (see Lemma 2.1 in [22]), so that we may assume, without any loss of generality, that $\mu_{\alpha,u}(x_1, \ldots, x_n) = 0$. Let then

$$\ell := \left\lceil \frac{n(1-\alpha)}{2} \right\rceil - 1$$

and fix an arbitrary d-variate sample $y_1, \ldots, y_n$ differing from $x_1, \ldots, x_n$ by at most ℓ observations. To keep the notation light, we denote the corresponding quantiles as $\mu^x_{\alpha,u}$ and $\mu^y_{\alpha,u}$. Since $\mu_{\alpha,u}$ is invariant under permutations of its arguments, we may assume that $y_i = x_i$ for all $i = \ell + 1, \ldots, n$. Define

$$d := \inf_{\mu \in B_{2M}} \|\mu^y_{\alpha,u} - \mu\|,$$

where we let $B_r := \{z \in \mathbb{R}^d : \|z\| \leq r\}$ and $M := \max_{i=\ell+1,\ldots,n} \|x_i\|$. For any $\varepsilon > 0$, there then exists $\mu \in B_{2M}$ such that

$$\|\mu^y_{\alpha,u}\| \leq \|\mu^y_{\alpha,u} - \mu\| + \|\mu\| \leq (d+\varepsilon) + 2M,$$

so that $\|\mu^y_{\alpha,u}\| \leq d + 2M$. We will show that

$$d \leq \frac{2M(\ell + n\alpha)}{n(1-\alpha) - 2\ell}. \tag{A.21}$$

Ad absurdum, assume that

$$d > \frac{2M(\ell + n\alpha)}{n(1-\alpha) - 2\ell}. \tag{A.22}$$

In particular, $d > 0$, so that, using that $\|y_i\| \leq M$ for $i = \ell+1, \ldots, n$, we have

$$\|y_i - \mu^y_{\alpha,u}\| \geq M + d \geq \|y_i\| + d$$

for $i = \ell+1, \ldots, n$. Now, since

$$\|y_i - \mu^y_{\alpha,u}\| \geq \|y_i\| - \|\mu^y_{\alpha,u}\| \geq \|y_i\| - (d + 2M),$$

for $i = 1, \ldots, n$, it follows that

$$\begin{aligned}
&\frac{1}{n}\sum_{i=1}^n \|y_i - \mu^y_{\alpha,u}\| - \alpha u' \mu^y_{\alpha,u} \\
&\qquad \geq \frac{1}{n}\sum_{i=1}^n \|y_i\| - \frac{\ell(d+2M)}{n} + \frac{(n-\ell)d}{n} - \alpha\|\mu^y_{\alpha,u}\| \\
&\qquad \geq \frac{1}{n}\sum_{i=1}^n \|y_i\| - \frac{(\ell+n\alpha)(d+2M)}{n} + \frac{(n-\ell)d}{n} \\
&\qquad = \frac{1}{n}\sum_{i=1}^n \|y_i\| + \frac{(n(1-\alpha)-2\ell)d - 2M(\ell+n\alpha)}{n} \\
&\qquad > \frac{1}{n}\sum_{i=1}^n \|y_i\|,
\end{aligned}$$

where the last inequality results from (A.22). Since this contradicts the fact that $\mu^y_{\alpha,u}$ minimizes $\mu \mapsto \frac{1}{n}\sum_{i=1}^n \|y_i - \mu\| - \alpha u'\mu$ over $\mathbb{R}^d$, this establishes (A.21), hence that

$$\|\mu^x_{\alpha,u} - \mu^y_{\alpha,u}\| = \|\mu^y_{\alpha,u}\| \leq d + 2M \leq \frac{2M(n-\ell)}{n(1-\alpha)-2\ell}.$$

Since the contaminated sample $y_1, \ldots, y_n$ was arbitrary, we have

$$\sup_y \|\mu^x_{\alpha,u} - \mu^y_{\alpha,u}\| \leq \frac{2M(n-\ell)}{n(1-\alpha)-2\ell} < \infty,$$

which shows that

$$\mathrm{BDP}(\mu_{\alpha,u}; x_1, \ldots, x_n) \geq \frac{\ell+1}{n},$$

hence concludes the proof. □

We turn to the proof of Theorem 2.2, which requires both following lemmas.

Lemma A.1. *For any* $v, w \in \mathbb{R}^d \setminus \{0\}$,

$$\left\| \frac{v}{\|v\|} - \frac{w}{\|w\|} \right\| \leq \frac{2\|v-w\|}{\|w\|}.$$

PROOF OF LEMMA A.1. Since the triangle inequality directly provides

$$\begin{aligned}
\left\| \frac{v}{\|v\|} - \frac{w}{\|w\|} \right\| &\leq \left\| \frac{\|w\|v - \|v\|v - \|v\|(w-v)}{\|v\|\|w\|} \right\| \\
&\qquad \leq \frac{\big|\|w\| - \|v\|\big|}{\|w\|} + \frac{\|w-v\|}{\|w\|} \leq 2\frac{\|v-w\|}{\|w\|},
\end{aligned}$$

the result is proved. □

Lemma A.2. *Fix $\alpha \in [0,1)$ and a positive integer n. Then, the integer ℓ in (2.4) satisfies $|n(1-\alpha)-\ell| < \ell$.*

PROOF OF LEMMA A.2. Since one has trivially $n(1-\alpha)-\ell > -\ell$, it is sufficient to show that $n(1-\alpha)-\ell < \ell$, that is, that

$$\ell > \frac{n(1-\alpha)}{2}.$$

Since this trivially follows from the definition of ℓ, the lemma is proved. □

We can now prove Theorem 2.2.

PROOF OF THEOREM 2.2. Recall that, for a general probability measure P, the objective function $M_{\alpha,u}(\mu;P)$ in (1.1) admits directional derivative in any direction $v(\in \mathcal{S}^{d-1})$, and that the corresponding directional derivative at μ is

$$\frac{\partial M_{\alpha,u}}{\partial v}(\mu;P) = (1-\alpha u'v)P[\{\mu\}] + v'\mathrm{E}\bigg[\bigg(\frac{\mu - Z}{\|\mu - Z\|} - \alpha u\bigg)\mathbb{I}[Z \neq \mu]\bigg];$$

see, e.g., Section 5 from [19]. For the empirical measure associated with the contaminated sample $y_1,\ldots,y_n$, this directional derivative at $\mu = ru$, for $r > M := \max_{i=\ell+1,\ldots,n}\|x_i\|$, is given by

$$\frac{\partial M^y_{\alpha,u}}{\partial v}(\mu) = (1-\alpha u'v)\frac{\ell}{n} + \frac{1}{n}\sum_{i=\ell+1}^{n}\frac{v'(ru-x_i)}{\|ru-x_i\|} - \frac{n-\ell}{n}\alpha v'u$$

$$= \frac{\ell}{n} + v'\bigg\{\bigg(\frac{n-\ell}{n}-\alpha\bigg)u + \frac{1}{n}\sum_{i=\ell+1}^{n}\bigg(\frac{ru-x_i}{\|ru-x_i\|} - u\bigg)\bigg\}. \tag{A.23}$$

Clearly, the minimal value of this directional derivative over the unit sphere is

$$\min_{v\in\mathcal{S}^{d-1}}\frac{\partial M^y_{\alpha,u}}{\partial v}(ru) = \frac{\ell}{n} - \bigg\|\bigg(\frac{n-\ell}{n}-\alpha\bigg)u + \frac{1}{n}\sum_{i=\ell+1}^{n}\bigg(\frac{ru-x_i}{\|ru-x_i\|} - u\bigg)\bigg\|.$$

Applying Lemma A.1, we have

$$\begin{aligned}
&\bigg\|\bigg(\frac{n-\ell}{n}-\alpha\bigg)u + \frac{1}{n}\sum_{i=\ell+1}^{n}\bigg(\frac{ru-x_i}{\|ru-x_i\|} - u\bigg)\bigg\| \\
&\qquad\le \bigg|\frac{n-\ell}{n}-\alpha\bigg| + \frac{1}{n}\sum_{i=\ell+1}^{n}\bigg\|\frac{ru-x_i}{\|ru-x_i\|} - \frac{ru}{\|ru\|}\bigg\| \\
&\qquad\le \frac{|n(1-\alpha)-\ell|}{n} + \frac{1}{n}\sum_{i=\ell+1}^{n}\frac{2\|x_i\|}{r} \\
&\qquad\le \frac{|n(1-\alpha)-\ell|}{n} + \frac{2(n-\ell)M}{nr}.
\end{aligned}$$

Therefore, using Lemma A.2, we have

$$\liminf_{r\to\infty}\min_{v\in\mathcal{S}^{d-1}}\frac{\partial M^y_{\alpha,u}}{\partial v}(ru) > 0.$$

Convexity of $M^y_{\alpha,u}$ then implies that $\mu_{\alpha,u}(y_1,\ldots,y_n)$ is uniquely defined and is equal to ru. □

The proof of Theorem 2.3 requires the following lemma.

Lemma A.3. *Let K be a bounded set of $\mathbb{R}^d$. Then, (i) for any $u \in \mathcal{S}^{d-1}$,*

$$\sup_{v\in\mathcal{S}^{d-1}}\sup_{x\in K}\bigg|v'u - \frac{v'(ru-x)}{\|ru-x\|} - \frac{v'(I_d-uu')x}{r}\bigg| = O\bigg(\frac{1}{r^2}\bigg)$$

as r diverges to infinity; (ii) for any $u \in \mathcal{S}^{d-1}$,

$$\sup_{x\in K}\left|1-\frac{u'(ru-x)}{\|ru-x\|}-\frac{\|x\|^2-(u'x)^2}{2r^2}\right|=O\left(\frac{1}{r^3}\right)$$

as r diverges to infinity.

Establishing Lemma A.3 in turn requires the following preliminary result.

Lemma A.4. *Let K be a bounded set of $\mathbb{R}^d$ and m be a non-negative integer. Then, any $u \in \mathcal{S}^{d-1}$, we have*

$$\sup_{x\in K}\left|\frac{1}{\|ru-x\|(\|ru-x\|+r)^m}-\frac{1}{2^m r^{m+1}}\right|=O\left(\frac{1}{r^{m+1}}\right)$$

as r diverges to infinity.

PROOF OF LEMMA A.4. Throughout the proof, C is a positive constant such that $\|x\| \leq C$ for any $x \in K$. Since, for r large enough, $\|ru-x\| \geq r/2$ for any $x \in K$, we have

$$\left|\frac{1}{\|ru-x\|}-\frac{1}{r}\right|=\left|\frac{r-\|ru-x\|}{r\|ru-x\|}\right|=\frac{|2r(u'x)-\|x\|^2|}{r\|ru-x\|(r+\|ru-x\|)}\leq\frac{2rC+C^2}{r(r/2)(r+(r/2))},$$

which establishes the result for $m=0$. Proceeding by induction, assume then that the result holds for $m-1$. We have (below, all O's are uniform in $x \in K$)

$$\begin{aligned}&\frac{1}{\|ru-x\|(\|ru-x\|+r)^m}-\frac{1}{2^m r^{m+1}}\\&\quad=\frac{1}{\|ru-x\|+r}\left(\frac{1}{2^{m-1}r^m}+O\left(\frac{1}{r^m}\right)\right)-\frac{1}{2^m r^{m+1}}\\&\quad=\frac{1}{2^{m-1}r^m}\left(\frac{1}{\|ru-x\|+r}-\frac{1}{2r}\right)+O\left(\frac{1}{r^{m+1}}\right),\end{aligned}$$

where

$$\left|\frac{1}{\|ru-x\|+r}-\frac{1}{2r}\right|=\left|\frac{r-\|ru-x\|}{\|ru-x\|+r}\right|=\left|\frac{2r(u'x)-\|x\|^2}{(\|ru-x\|+r)^2}\right|\leq\frac{2rC+C^2}{((r/2)+r)^2}.$$

We conclude that

$$\sup_{x\in K}\left|\frac{1}{\|ru-x\|(\|ru-x\|+r)^m}-\frac{1}{2^m r^{m+1}}\right|=\frac{1}{2^{m-1}r^m}O\left(\frac{1}{r}\right)+O\left(\frac{1}{r^{m+1}}\right),$$

which establishes the result. □

We can now prove Lemma A.3.

PROOF OF LEMMA A.3. Throughout the proof, C is still a positive constant such that $\|x\| \leq C$ for any $x \in K$, and we assume that r is large enough to have $\|ru-x\| \geq r/2$ for any $x \in K$. (i) Since

$$\begin{aligned}v'u-\frac{v'(ru-x)}{\|ru-x\|}&=\frac{(u'v)\|ru-x\|-v'(ru-x)}{\|ru-x\|}\\&=\frac{(u'v)\|ru-x\|-r(u'v)}{\|ru-x\|}+\frac{r(u'v)-v'(ru-x)}{\|ru-x\|}\\&=\frac{\|ru-x\|-r}{\|ru-x\|}(u'v)+\frac{x'v}{\|ru-x\|}\end{aligned}$$

$$= \frac{\|x\|^2 - 2r(u'x)}{\|ru-x\|(\|ru-x\|+r)}(u'v) + \frac{x'v}{\|ru-x\|},$$

we have

$$v'u - \frac{v'(ru-x)}{\|ru-x\|} - \frac{v'(I_d - uu')x}{r} = S_1 + S_2 + S_3,$$

with

$$S_1 := \left(\frac{1}{\|ru-x\|} - \frac{1}{r}\right)(x'v),$$

$$S_2 := \left(\frac{-2r}{\|ru-x\|(\|ru-x\|+r)} + \frac{1}{r}\right)(u'x)(u'v)$$

and

$$S_3 := \frac{\|x\|^2}{\|ru-x\|(\|ru-x\|+r)}(u'v)\cdot$$

Since

$$\left|\frac{1}{\|ru-x\|} - \frac{1}{r}\right| = \left|\frac{r - \|ru-x\|}{r\|ru-x\|}\right| = \left|\frac{2r(u'x) - \|x\|^2}{r\|ru-x\|(r+\|ru-x\|)}\right| \leq \frac{2rC + C^2}{r(r/2)(r+(r/2))},$$

we have

$$\sup_{v\in\mathcal{S}^{d-1}} \sup_{x\in K} |S_1| = O\left(\frac{1}{r^2}\right).$$

Since Lemma A.4 directly yields

$$\sup_{v\in\mathcal{S}^{d-1}} \sup_{x\in K} |S_2| = O\left(\frac{1}{r^2}\right)$$

and

$$|S_3| \leq \frac{C^2}{(r/2)((r/2)+r)} = O\left(\frac{1}{r^2}\right),$$

Part (i) of the result is proved.

(ii) Since

$$\begin{aligned}
1 - \frac{u'(ru-x)}{\|ru-x\|} &= \frac{1}{\|ru-x\|}(\|ru-x\| - r + u'x) \\
&= \frac{1}{\|ru-x\|}\left(\frac{\|x\|^2 - 2r(u'x)}{\|ru-x\|+r} + u'x\right) \\
&= \frac{\|x\|^2}{\|ru-x\|(\|ru-x\|+r)} + \frac{(\|ru-x\|+r)(u'x) - 2r(u'x)}{\|ru-x\|(\|ru-x\|+r)} \\
&= \frac{\|x\|^2}{\|ru-x\|(\|ru-x\|+r)} + \frac{\|ru-x\|(u'x) - r(u'x)}{\|ru-x\|(\|ru-x\|+r)},
\end{aligned}$$

we have

$$1 - \frac{u'(ru-x)}{\|ru-x\|} - \frac{\|x\|^2 - (u'x)^2}{2r^2} = T_1 + T_2,$$

with

$$T_1 := \left(\frac{1}{\|ru-x\|(\|ru-x\|+r)} - \frac{1}{2r^2}\right)\|x\|^2$$

and

$$T_2 := \left(\frac{\|ru - x\| - r}{\|ru - x\|(\|ru - x\| + r)} + \frac{u'x}{2r^2} \right)(u'x)$$

$$= \left(\frac{\|x\|^2 - 2r(u'x)}{\|ru - x\|(\|ru - x\| + r)^2} + \frac{u'x}{2r^2} \right)(u'x)$$

$$= \frac{\|x\|^2(u'x)}{\|ru - x\|(\|ru - x\| + r)^2} + \left(\frac{-2r}{\|ru - x\|(\|ru - x\| + r)^2} + \frac{1}{2r^2} \right)(u'x)^2$$

$$=: T_{2a} + T_{2b},$$

say. Since Lemma A.4 directly yields

$$\sup_{v \in \mathcal{S}^{d-1}} \sup_{x \in K} |T_1| = O\left(\frac{1}{r^2}\right) \quad \text{and} \quad \sup_{v \in \mathcal{S}^{d-1}} \sup_{x \in K} |T_{2b}| = O\left(\frac{1}{r^2}\right)$$

and

$$|T_{2a}| \leq \frac{C^3}{(r/2)((r/2) + r)^2},$$

the result is proved. □

We can finally prove Theorem 2.3.

PROOF OF THEOREM 2.3. First note that (A.23), with $m + ru$ instead of ru, here yields

$$\frac{\partial M^y_{\alpha,u}}{\partial v}(m + ru) = \frac{\ell}{n} - \alpha v'u + \frac{1}{n} \sum_{i=\ell+1}^{n} \frac{v'(m + ru - x_i)}{\|m + ru - x_i\|}$$

$$= \frac{\ell}{n}(1 + u'v) + \frac{n(1-\alpha) - 2\ell}{n} u'v + \frac{1}{n} \sum_{i=\ell+1}^{n} \left(\frac{v'(m + ru - x_i)}{\|m + ru - x_i\|} - u'v \right)$$

$$= \frac{\ell}{n} + v' \left\{ \frac{\ell}{n} u + \frac{1}{n} \sum_{i=\ell+1}^{n} \left(\frac{m + ru - x_i}{\|m + ru - x_i\|} - u \right) \right\}.$$

The minimal directional derivative on the unit sphere is then

$$\min_{v \in \mathcal{S}^{d-1}} \frac{\partial M^y_{\alpha,u}}{\partial v}(m + ru) = \frac{\ell}{n} - \left\| \frac{\ell}{n} u + \frac{1}{n} \sum_{i=\ell+1}^{n} \left(\frac{m + ru - x_i}{\|m + ru - x_i\|} - u \right) \right\|.$$

Since Lemma A.1 entails that

$$\left\{ \frac{\ell}{n} + \left\| \frac{\ell}{n} u + \frac{1}{n} \sum_{i=\ell+1}^{n} \left(\frac{m + ru - x_i}{\|m + ru - x_i\|} - u \right) \right\| \right\} = \frac{2\ell}{n} + O\left(\frac{1}{r}\right)$$

as r diverges to infinity,

$$\left(\frac{2\ell}{n} + O\left(\frac{1}{r}\right) \right) \min_{v \in \mathcal{S}^{d-1}} \frac{\partial M^y_{\alpha,u}}{\partial v}(m + ru) = -T_1(r) + T_2(r), \tag{A.24}$$

where we let

$$T_1(r) := \left\| \frac{1}{n} \sum_{i=\ell+1}^{n} \left(\frac{m + ru - x_i}{\|m + ru - x_i\|} - u \right) \right\|^2$$

and

$$T_2(r) := -\frac{2\ell}{n^2} \sum_{i=\ell+1}^{n} \left(\frac{u'(m+ru-x_i)}{\|m+ru-x_i\|} - 1 \right).$$

Lemma A.3(i) entails that

$$\sup_{v \in \mathcal{S}^{d-1}} \left| v' \left\{ \frac{1}{n} \sum_{i=\ell+1}^{n} \left(\frac{m+ru-x_i}{\|m+ru-x_i\|} - u \right) \right\} + \frac{1}{nr} \sum_{i=\ell+1}^{n} v'(I_d - uu')(x_i - m) \right| = O\left(\frac{1}{r^2}\right),$$

so that the definition of m ensures that

$$T_1(r) = O\left(\frac{1}{r^4}\right).$$

Since Lemma A.3(ii) provides

$$T_2(r) = \frac{\ell}{n^2 r^2} \sum_{i=\ell+1}^{n} \left(\|x_i - m\|^2 - (u'(x_i - m))^2 \right) + O\left(\frac{1}{r^3}\right),$$

we conclude from (A.24) that

$$\min_{v \in \mathcal{S}^{d-1}} \frac{\partial M^y_{\alpha,u}}{\partial v}(m+ru) = \frac{1}{2nr^2} \sum_{i=\ell+1}^{n} \left(\|x_i - m\|^2 - (u'(x_i - m))^2 \right) + O\left(\frac{1}{r^3}\right). \tag{A.25}$$

We may now consider both cases in the statement of the theorem. (i) If it is not so that $x_{\ell+1}, \ldots, x_n$ all belong to the line $\mathcal{L} = \{m + \lambda u : \lambda \in \mathbb{R}\}$, then (A.25) ensures that there exists $R > 0$ such that

$$\min_{v \in \mathcal{S}^{d-1}} \frac{\partial M^y_{\alpha,u}}{\partial v}(m+ru) > 0$$

for any $r \geq R$, which, from the convexity of the objective function defining spatial quantiles, implies that, for $r \geq R$, $\mu_{\alpha,u}(y_1, \ldots, y_n)$ is uniquely defined and is equal to $m + ru$. (ii) If $x_{\ell+1}, \ldots, x_n$ all belong to $\mathcal{L}$, then $y_1, \ldots, y_n$ all belong to $\mathcal{L}$ and take the form $m+ru, \ldots, m+ru, m+\lambda_{\ell+1}u, \ldots, m+\lambda_n u$, with $\lambda_i = u'(x_i - m)$. From Lemma S.1.1 in [20], any spatial quantile of order α in direction u for the empirical probability measure associated with $y_1, \ldots, y_n$ then itself belongs to $\mathcal{L}$, and, if $r > \max_{i=\ell+1,\ldots,n} u'(x_i - m)$, then a direct computation shows that the collection of these quantiles is given by $[\max_{i=\ell+1,\ldots,n} \lambda_i, r]$. By definition, we thus have

$$\mu_{\alpha,u}(y_1, \ldots, y_n) = m + \frac{(\max_{i=\ell+1,\ldots,n} \lambda_i) + r}{2} u,$$

which establishes the result. □

Appendix B: Proofs for Section 3

Before proving Theorem 3.1, we note that when establishing results on spatial quantiles, we may safely substitute the objective function

$$\mu \mapsto \tilde{M}_{\alpha,u}(\mu; x_1, \ldots, x_n) := \frac{1}{n} \sum_{i=1}^{n} \|x_i - \mu\| + \frac{\alpha}{n} \sum_{i=1}^{n} u'(x_i - \mu) \tag{B.26}$$

for the original one

$$\mu \mapsto M_{\alpha,u}(\mu; x_1, \ldots, x_n) = \frac{1}{n} \sum_{i=1}^{n} (\|x_i - \mu\| - \|x_i\|) - \alpha u'\mu$$

in (1.2), as these objective functions share the same minimizers.

PROOF OF THEOREM 3.1. We consider the objective function in (B.26), which, at the contaminated sample $y_1, \ldots, y_n$ defined in the statement of the theorem, takes the value

$$\tilde{M}_{\alpha,u}(\mu; y_1, \ldots, y_n) = \frac{\ell}{n}(\|rv - \mu\| + \alpha u'(rv - \mu)) + \frac{1}{n}\sum_{i=\ell+1}^{n} \|x_i - \mu\| + \frac{\alpha}{n}\sum_{i=\ell+1}^{n} u'(x_i - \mu).$$

At $\mu = rv$, we have

$$\tilde{M}_{\alpha,u}(\mu; y_1, \ldots, y_n) = \frac{1}{n}\sum_{i=\ell+1}^{n} \|x_i - \mu\| + \frac{\alpha}{n}\sum_{i=\ell+1}^{n} u'(x_i - \mu) = \frac{(n-\ell)(1-\alpha u'v)}{n} r + o(r),$$

as r diverges to infinity. Now,

$$\begin{aligned}
&\tilde{M}_{\alpha,u}(\mu; y_1, \ldots, y_n) \\
&\geq \frac{\ell}{n}\|rv - \mu\| + \frac{\ell\alpha}{n} u'(rv - \mu) \\
&= \frac{\ell}{n}\|rv - \mu\| \left(1 + \alpha \frac{u'(rv - \mu)}{\|rv - \mu\|}\right) \\
&= \frac{\ell}{n}\|rv - \mu\|(1 + \alpha u'v) + \frac{\ell\alpha}{n}\|rv - \mu\| \left(\frac{u'(v - \mu/r)}{\|v - \mu/r\|} - \frac{u'v}{\|v\|}\right) \\
&\geq \frac{\ell}{n}\|rv - \mu\|(1 + \alpha u'v) - \frac{\ell\alpha}{n}\|rv - \mu\| \left\| \frac{v - \mu/r}{\|v - \mu/r\|} - \frac{v}{\|v\|} \right\| \\
&\geq \frac{\ell}{n}\|rv - \mu\|(1 + \alpha u'v) - \frac{2\ell\alpha}{nr}\|rv - \mu\| \frac{\|\mu\|}{\|v\|},
\end{aligned}$$

so that, for any $\mu \in B(\sqrt{r})$ with $r > 1$, we have

$$\tilde{M}_{\alpha,u}(\mu; y_1, \ldots, y_n) \geq \frac{\ell(1 + \alpha u'v)}{n}(r - \sqrt{r}) - \frac{2\ell\alpha}{n\sqrt{r}}(r + \sqrt{r}) = \frac{\ell(1 + \alpha u'v)}{n}(r - \sqrt{r}) + o(r).$$

Thus, for any net $(\mu_r)_{r>0}$ with $\|\mu_r\| \leq \sqrt{r}$ for all $r > 0$, we have

$$\liminf_{r\to\infty} \frac{\tilde{M}_{\alpha,u}(rv; y_1, \ldots, y_n)}{\tilde{M}_{\alpha,u}(\mu_r; y_1, \ldots, y_n)} \leq \frac{(n-\ell)(1-\alpha u'v)}{\ell(1+\alpha u'v)} = 1 + \frac{n(1-\alpha u'v) - 2\ell}{\ell(1+\alpha u'v)} \cdot$$

Since $2\ell > n(1 - \alpha u'v)$, this entails that there exists $\varepsilon \in (0, 1)$ such that

$$\tilde{M}_{\alpha,u}(rv; y_1, \ldots, y_n) < (1-\varepsilon)\tilde{M}_{\alpha,u}(\mu_r; y_1, \ldots, y_n)$$

for r large enough and any net (μ_r) with $\|\mu_r\| \leq \sqrt{r}$ for all r. Now, choose a net (μ_r) of this type such that

$$\tilde{M}_{\alpha,u}(\mu_r; y_1, \ldots, y_n) \leq \frac{1}{1-\varepsilon} \inf_{\mu \in B(\sqrt{r})} \tilde{M}_{\alpha,u}(\mu; y_1, \ldots, y_n).$$

We then have for r large enough

$$\tilde{M}_{\alpha,u}(rv; y_1, \ldots, y_n) < \inf_{\mu \in B(\sqrt{r})} \tilde{M}_{\alpha,u}(\mu; y_1, \ldots, y_n).$$

This proves that $\mu_{\alpha,u}(y_1, \ldots, y_n)$ does not belong to $B(\sqrt{r})$ for r large enough, which establishes the result. □

In Section 3, it only remains to prove Theorem 3.2. Sample L_p-quantiles are defined through the minimizers of the objective function

$$\mu \mapsto M_{\alpha,u,p}(\mu; x_1, \ldots, x_n) = \frac{1}{n} \sum_{i=1}^{n} (H_{\alpha,u,p}(x_i - \mu) - H_{\alpha,u,p}(x_i)),$$

with

$$H_{\alpha,u,p}(z) = \|z\|^p \left(1 + \alpha \frac{u'z}{\|z\|}\right) \mathbb{I}[z \neq 0],$$

but we may of course safely rather consider the objective function

$$\mu \mapsto \tilde{M}_{\alpha,u,p}(\mu; x_1, \ldots, x_n) = \frac{1}{n} \sum_{i=1}^{n} \|x_i - \mu\|^p + \frac{\alpha}{n} \sum_{i=1}^{n} \|x_i - \mu\|^{p-1} u'(x_i - \mu), \tag{B.27}$$

as it admits the same minimizers.

PROOF OF THEOREM 3.2. At the contaminated sample $y_1, \ldots, y_n$ defined in the statement of the result, the objective function in (B.27) writes

$$\tilde{M}_{\alpha,u,p}(\mu; y_1, \ldots, y_n) = \frac{1}{n} \|rv - \mu\|^p + \frac{\alpha}{n} \|rv - \mu\|^{p-1} u'(rv - \mu)$$
$$+ \frac{1}{n} \sum_{i=2}^{n} \|x_i - \mu\|^p + \frac{\alpha}{n} \sum_{i=2}^{n} \|x_i - \mu\|^{p-1} u'(x_i - \mu).$$

Let $c \in (0, 1)$ be fixed (the value of c will be chosen later). At $\mu = crv$, we have

$$\begin{aligned}
\tilde{M}_{\alpha,u,p}(\mu; y_1, \ldots, y_n) &= \frac{1 + \alpha u'v}{n} (1-c)^p r^p \\
&\qquad + \frac{1}{n} \sum_{i=2}^{n} \|x_i - \mu\|^p + \frac{\alpha}{n} \sum_{i=2}^{n} \|x_i - \mu\|^{p-1} u'(x_i - \mu) \\
&= \frac{1 + \alpha u'v}{n} (1-c)^p r^p + \frac{(1 - \alpha u'v)(n-1)}{n} c^p r^p + o(r^p),
\end{aligned}$$

as r diverges to infinity. Now,

$$\begin{aligned}
&\tilde{M}_{\alpha,u,p}(\mu; y_1, \ldots, y_n) \\
&\geq \frac{1}{n} \|rv - \mu\|^p + \frac{\alpha}{n} \|rv - \mu\|^{p-1} u'(rv - \mu) \\
&= \frac{1}{n} \|rv - \mu\|^p \left(1 + \alpha \frac{u'(rv - \mu)}{\|rv - \mu\|}\right) \\
&= \frac{1}{n} \|rv - \mu\|^p (1 + \alpha u'v) + \frac{\alpha}{n} \|rv - \mu\|^p \left(\frac{u'(v - \mu/r)}{\|v - \mu/r\|} - \frac{u'v}{\|v\|}\right)
\end{aligned}$$

$$\geq \frac{1}{n}\|rv-\mu\|^p(1+\alpha u'v) - \frac{\alpha}{n}\|rv-\mu\|^p \left\| \frac{v-\mu/r}{\|v-\mu/r\|} - \frac{v}{\|v\|} \right\|$$

$$\geq \frac{1}{n}\|rv-\mu\|^p(1+\alpha u'v) - \frac{2\alpha}{nr}\|rv-\mu\|^p \frac{\|\mu\|}{\|v\|},$$

so that, for any $\mu \in B(\sqrt{r})$ with r large enough to have $\sqrt{r} < cr$, we have

$$\tilde{M}_{\alpha,u,p}(\mu; y_1,\ldots,y_n) \geq \frac{1+\alpha u'v}{n}(r-\sqrt{r})^p - \frac{2\alpha}{n\sqrt{r}}(r+\sqrt{r})^p$$

$$= \frac{1+\alpha u'v}{n}(r-\sqrt{r})^p + o(r^p).$$

Thus, reasoning as in the proof of Theorem 3.1 yields

$$\liminf_{r\to\infty} \frac{\tilde{M}_{\alpha,u,p}(crv; y_1,\ldots,y_n)}{\inf_{\mu\in B(\sqrt{r})} \tilde{M}_{\alpha,u,p}(\mu; y_1,\ldots,y_n)} \leq (1-c)^p + \frac{(1-\alpha u'v)(n-1)}{1+\alpha u'v} c^p =: f(c).$$

Since $f(0)=1$ and $f'(0)<0$, it is possible to pick $c\in(0,1)$ such that $f(c)<1$. With this value of c, we have thus showed that

$$\tilde{M}_{\alpha,u,p}(crv; y_1,\ldots,y_n) < \inf_{\mu\in B(\sqrt{r})} \tilde{M}_{\alpha,u,p}(\mu; y_1,\ldots,y_n)$$

for r large enough, which proves that the L_p-quantile of order α in direction u for $y_1,\ldots,y_n$ does not belong to $B(\sqrt{r})$, hence diverges to infinity in norm. □

We now proceed with the proof of Theorem 3.3.

PROOF OF THEOREM 3.3. At the contaminated probability measure P_n, the objective function in (B.27) writes

$$\tilde{M}_{\alpha,u,p}(\mu; P_n) = \frac{1}{2n}\|rv-\mu\|^p + \frac{\alpha}{2n}\|rv-\mu\|^{p-1}u'(rv-\mu) + \frac{1}{2n}\|-rv-\mu\|^p + \frac{\alpha}{2n}\|-rv-\mu\|^{p-1}u'(-rv-\mu)$$

$$+\frac{1}{n}\sum_{i=2}^{n}\|x_i-\mu\|^p + \frac{\alpha}{n}\sum_{i=2}^{n}\|x_i-\mu\|^{p-1}u'(x_i-\mu).$$

Let $c\in(0,1)$ be a fixed constant, to be determined later in the proof. At $\mu = cru$, we have

$$\tilde{M}_{\alpha,u,p}(\mu; P_n) = \frac{1}{2n}\|v-cu\|^p r^p + \frac{\alpha}{2n}\|v-cu\|^{p-1}r^p u'(v-cu) + \frac{1}{2n}\|v+cu\|^p r^p - \frac{\alpha}{2n}\|v+cu\|^{p-1}r^p u'(v+cu)$$

$$+\frac{1}{n}\sum_{i=2}^{n}\|x_i-cru\|^p + \frac{\alpha}{n}\sum_{i=2}^{n}\|x_i-cru\|^{p-1}u'(x_i-cru)$$

$$= \frac{1}{2n}\|v-cu\|^p r^p + \frac{\alpha}{2n}\|v-cu\|^{p-1}r^p u'(v-cu) + \frac{1}{2n}\|v+cu\|^p r^p - \frac{\alpha}{2n}\|v+cu\|^{p-1}r^p u'(v+cu)$$

$$+\frac{(1-\alpha)(n-1)}{n}c^p r^p + o(r^p),$$

as r diverges to infinity. Now,

$$\tilde{M}_{\alpha,u,p}(\mu; P_n)$$

$$\geq \frac{1}{2n}\|rv-\mu\|^p + \frac{\alpha}{2n}\|rv-\mu\|^{p-1}u'(rv-\mu) + \frac{1}{2n}\|-rv-\mu\|^p + \frac{\alpha}{2n}\|-rv-\mu\|^{p-1}u'(-rv-\mu)$$

$$= \frac{1}{2n}\|rv-\mu\|^p\left(1+\alpha\frac{u'(rv-\mu)}{\|rv-\mu\|}\right) + \frac{1}{2n}\|rv+\mu\|^p\left(1-\alpha\frac{u'(rv+\mu)}{\|rv+\mu\|}\right)$$

$$= \frac{1}{2n}\|rv-\mu\|^p(1+\alpha u'v) + \frac{\alpha}{2n}\|rv-\mu\|^p\left(\frac{u'(v-\mu/r)}{\|v-\mu/r\|} - \frac{u'v}{\|v\|}\right)$$

$$+\frac{1}{2n}\|rv+\mu\|^p(1-\alpha u'v) - \frac{\alpha}{2n}\|rv+\mu\|^p\left(\frac{u'(v+\mu/r)}{\|v+\mu/r\|} - \frac{u'v}{\|v\|}\right)$$

$$\geq \frac{1}{2n}\|rv-\mu\|^p(1+\alpha u'v) - \frac{\alpha}{2n}\|rv-\mu\|^p\left\|\frac{v-\mu/r}{\|v-\mu/r\|} - \frac{v}{\|v\|}\right\|$$

$$+\frac{1}{2n}\|rv+\mu\|^p(1-\alpha u'v) - \frac{\alpha}{2n}\|rv+\mu\|^p\left\|\frac{v+\mu/r}{\|v+\mu/r\|} - \frac{v}{\|v\|}\right\|$$

$$\geq \frac{1}{2n}\|rv-\mu\|^p(1+\alpha u'v) + \frac{1}{2n}\|rv+\mu\|^p(1-\alpha u'v) - \frac{\alpha}{nr}\|rv-\mu\|^p\frac{\|\mu\|}{\|v\|} - \frac{\alpha}{nr}\|rv+\mu\|^p\frac{\|\mu\|}{\|v\|},$$

so that, for any $\mu \in B(\sqrt{r})$ with r large enough to have $\sqrt{r} < cr$, we have

$$\tilde{M}_{\alpha,u,p}(\mu;P_n) \geq \frac{1+\alpha u'v}{2n}(r-\sqrt{r})^p + \frac{1-\alpha u'v}{2n}(r-\sqrt{r})^p - \frac{\alpha}{n\sqrt{r}}(r+\sqrt{r})^p$$

$$= \frac{1}{n}(r-\sqrt{r})^p + o(r^p).$$

Thus, reasoning as in the proof of Theorem 3.1 yields

$$\liminf_{r\to\infty} \frac{\tilde{M}_{\alpha,u,p}(cru;P_n)}{\inf_{\mu\in B(\sqrt{r})}\tilde{M}_{\alpha,u,p}(\mu;P_n)} \leq \frac{1}{2}\|v-cu\|^p + \frac{\alpha}{2}\|v-cu\|^{p-1}(u'v-c)$$

$$+\frac{1}{2}\|v+cu\|^p - \frac{\alpha}{2}\|v+cu\|^{p-1}(u'v+c) + (1-\alpha)(n-1)c^p =: f(c).$$

Since $f(0) = 1$ and

$$f'(0) = -\alpha(p-1)(u'v)^2 - \alpha < 0,$$

it is possible to pick $c \in (0,1)$ such that $f(c) < 1$. With this value of c, we have thus showed that

$$\tilde{M}_{\alpha,u,p}(cru;P_n) < \inf_{\mu\in B(\sqrt{r})}\tilde{M}_{\alpha,u,p}(\mu;P_n)$$

for r large enough, which proves that the L_p-quantile of order α in direction u for P_n does not belong to $B(\sqrt{r})$, hence diverges to infinity in norm. □

Appendix C: Proofs for Section 4

The proof of Theorem 4.1 requires Lemmas C.1–C.4 below.

Lemma C.1. *Fix $\mu \in \mathbb{R}^d$. Then,*

$$\left|\|z-\mu\| - \|z\| + \frac{\mu'z}{\|z\|}\right| \leq 2\min\left(\frac{\|\mu\|^2}{\|z\|}, \|\mu\|\right) \tag{C.28}$$

for any $z \in \mathbb{R}^d \setminus \{0\}$.

PROOF OF LEMMA C.1. Fix $z \in \mathbb{R}^d \setminus \mathcal{D}$, with $\mathcal{D} := \{t\mu : t \in [0,1]\}$. Since the function $t \mapsto f(t) = \|z - t\mu\|$ is then continuous on $[0,1]$ and differentiable on $(0,1)$, the mean-value theorem ensures that there exists $t \in (0,1)$ such that

$$\|z-\mu\| - \|z\| + \frac{z'\mu}{\|z\|} = f(1) - f(0) + \frac{z'\mu}{\|z\|} = \frac{\mu'z}{\|z\|} - \frac{\mu'(z-t\mu)}{\|z-t\mu\|}.$$

By applying Lemma A.1, this yields

$$\left| \|z-\mu\| - \|z\| + \frac{\mu' z}{\|z\|} \right| \le \|\mu\| \left\| \frac{z}{\|z\|} - \frac{z-t\mu}{\|z-t\mu\|} \right\| \le \frac{2\|\mu\|^2}{\|z\|}.$$

Since we also trivially have

$$\left| \|z-\mu\| - \|z\| + \frac{\mu' z}{\|z\|} \right| \le \big| \|z-\mu\| - \|z\| \big| + \frac{|\mu' z|}{\|z\|} \le 2\|\mu\|,$$

we proved that (C.28) holds for any $z \in \mathbb{R}^d \setminus \mathcal{D}$. The result then follows from continuity. □

The next lemma, which we state under the assumptions of Theorem 4.1, actually does not require any assumption on the sequence (Q_k). Throughout, we let B_r stand for the open ball of radius $r(>0)$ centered at the origin of $\mathbb{R}^d$.

Lemma C.2. *Under the assumptions of Theorem 4.1, we have*

$$\sup_{\mu \in B_R} \left| M^{P_k}_{\alpha,u}(\mu) - (1-c) M^{Q_k}_{\tilde\alpha,\tilde u}(\mu) \right| \le 2R \Big(\|w_k - w\| + \Lambda_k(B_R) + \frac{R}{T} + \Lambda_k(B_T) \Big), \tag{C.29}$$

for any $T > R > 0$ and any $k \in \mathbb{N}$. In particular,

$$\sup_{\mu \in B_R} \left| M^{P_k}_{\alpha,u}(\mu) - (1-c) M^{Q_k}_{\tilde\alpha,\tilde u}(\mu) \right| \to 0 \tag{C.30}$$

for any $R > 0$ as $k \to \infty$.

Proof of Lemma C.2. Fix $R > 0$, $\mu \in B_R$, and $k \in \mathbb{N}$. We have

$$\begin{aligned} M^{P_k}_{\alpha,u}(\mu) &= (1-c) M^{Q_k}_{\alpha,u}(\mu) + c M^{\Lambda_k}_{\alpha,u}(\mu) \\ &= (1-c) \int_{\mathbb{R}^d} \Big(\|z-\mu\| - \|z\| - \alpha u' \mu \Big) \, dQ_k(z) \\ &\qquad\qquad + c \int_{\mathbb{R}^d} \Big(\|z-\mu\| - \|z\| - \alpha u' \mu \Big) \, d\Lambda_k(z) \\ &= (1-c) \int_{\mathbb{R}^d} \Big(\|z-\mu\| - \|z\| \Big) \, dQ_k(z) - \alpha u' \mu \\ &\qquad\qquad + c \int_{\mathbb{R}^d} \Big(\|z-\mu\| - \|z\| \Big) \, d\Lambda_k(z). \end{aligned}$$

This allows us to write

$$M^{P_k}_{\alpha,u}(\mu) = (1-c) M^{Q_k}_{\tilde\alpha,\tilde u}(\mu) + c w' \mu + c \int_{\mathbb{R}^d} \Big(\|z-\mu\| - \|z\| \Big) \, d\Lambda_k(z).$$

Lemma C.1 entails that

$$\begin{aligned} &\left| w' \mu + \int_{\mathbb{R}^d} \Big(\|z-\mu\| - \|z\| \Big) \, d\Lambda_k(z) \right| \\ &\le \left| \mu' w - \int_{\mathbb{R}^d \setminus \{0\}} \frac{\mu' z}{\|z\|} \, d\Lambda_k(z) \right| + 2 \int_{\mathbb{R}^d \setminus \{0\}} \min\left(\frac{\|\mu\|^2}{\|z\|}, \|\mu\| \right) d\Lambda_k(z) + \|\mu\| \Lambda_k(\{0\}) \\ &\le \|\mu\| \left\{ \|w_k - w\| + 2 \int_{\mathbb{R}^d \setminus \{0\}} \min\left(\frac{\|\mu\|}{\|z\|}, 1 \right) d\Lambda_k(z) + \Lambda_k(\{0\}) \right\}. \end{aligned}$$

Observe that

$$\int_{\mathbb{R}^d\setminus\{0\}} \min\left(\frac{\|\mu\|}{\|z\|},1\right) d\Lambda_k(z) \le \Lambda_k(B_{\|\mu\|}\setminus\{0\}) + \|\mu\| \int_{\mathbb{R}^d\setminus B_{\|\mu\|}} \frac{1}{\|z\|}\, d\Lambda_k(z).$$

Fix an arbitrary $T > R$. Then,

$$\int_{\mathbb{R}^d\setminus B_{\|\mu\|}} \frac{1}{\|z\|}\, d\Lambda_k(z) = \int_{\mathbb{R}^d\setminus B_T} \frac{1}{\|z\|}\, d\Lambda_k(z) + \int_{B_T\setminus B_{\|\mu\|}} \frac{1}{\|z\|}\, d\Lambda_k(z) \le \frac{1}{T} + \frac{\Lambda_k(B_T)}{\|\mu\|}\cdot$$

This implies that

$$\int_{\mathbb{R}^d\setminus\{0\}} \min\left(\frac{\|\mu\|}{\|z\|},1\right) d\Lambda_k(z) \le \Lambda_k(B_R\setminus\{0\}) + \frac{R}{T} + \Lambda_k(B_T),$$

for any $\mu \in B_R$, $T > R$ and k. Consequently, we have

$$\sup_{\mu\in B_R} \left| w'\mu + \int_{\mathbb{R}^d} \Big(\|z-\mu\| - \|z\|\Big)\, d\Lambda_k(z) \right| \le 2R\bigg(\|w_k - w\| + \Lambda_k(B_R) + \frac{R}{T} + \Lambda_k(B_T) \bigg),$$

hence also (C.29), for any $T > R$ and k. Now, it follows from (C.29) that

$$\limsup_{k\to\infty} \sup_{\mu\in B_R} \left| M^{P_k}_{\alpha,u}(\mu) - (1-c) M^{Q_k}_{\tilde{\alpha},\tilde{u}}(\mu) \right| \le \frac{2R^2}{T}\cdot$$

Because T was arbitrary, this establishes (C.30). □

Lemma C.3. *Let (Q_k) be a sequence of probability measures on $\mathbb{R}^d$. Fix $\beta \in \mathbb{R}$ and $u \in \mathcal{S}^{d-1}$. If (Q_k) converges weakly to a probability measure Q, then*

$$\sup_{\mu\in B_R} \left| M^{Q_k}_{\beta,u}(\mu) - M^{Q}_{\beta,u}(\mu) \right| \to 0 \tag{C.31}$$

for any $R > 0$ as $k \to \infty$.

PROOF OF LEMMA C.3. Fix $R > 0$ and write

$$M^{Q_k}_{\beta,u}(\mu) - M^{Q}_{\beta,u}(\mu) = \int_{\mathbb{R}^d} g_\mu(z)\, dQ_k(z) - \int_{\mathbb{R}^d} g_\mu(z)\, dQ(z),$$

for all $\mu \in \mathbb{R}^d$, where we let $g_\mu(z) = \|z-\mu\| - \|z\|$. The class of functions

$$\Gamma_R = \{g_\mu : \mu \in B_R\} \subset C^0(\mathbb{R}^d)$$

is uniformly bounded by R. Further observe that, for any $\mu \in \mathbb{R}^d$, we have

$$|g_\mu(z) - g_\mu(z')| \le 2\|z - z'\|$$

for all $z, z' \in \mathbb{R}^d$. It follows that Γ_R is equicontinuous at every point: for any $z \in \mathbb{R}^d$ and $\varepsilon > 0$, there exists $\delta > 0$ such that $|g_\mu(z) - g_\mu(z')| < \varepsilon$ for all $z' \in B_\delta(z)$ and $\mu \in B_R$. Therefore, Theorem 2.2.8 from [2] entails that

$$\sup_{f\in\Gamma_R} \left| \int_{\mathbb{R}^d} f(z)\, dQ_k(z) - \int_{\mathbb{R}^d} f(z)\, dQ(z) \right| \to 0$$

as $k \to \infty$. This yields

$$\sup_{\mu\in B_R} \left| M^{Q_k}_{\beta,u}(\mu) - M^{Q}_{\beta,u}(\mu) \right| \to 0$$

as $k \to \infty$. □

The next result is a refinement of Lemma 5 in [27].

Lemma C.4. *Let P be a probability measure on $\mathbb{R}^d$ and fix $u \in \mathcal{S}^{d-1}$. The objective function $M^P_{1,u}$ admits a minimum over $\mathbb{R}^d$ if and only if P is supported on a halfline with direction $-u$, say, the halfline $\{z_0 - \lambda u : \lambda \geq 0\}$. In this case, any $\mu = z_0 + \lambda u$, with $\lambda \geq 0$, minimizes $M^P_{1,u}$ over $\mathbb{R}^d$.*

PROOF OF LEMMA C.4. We first show that the infimum value of $M^P_{1,u}$ over $\mathbb{R}^d$ is given by

$$m^* := -\int_{\mathbb{R}^d} (\|z\| + u'z)\, dP(z).$$

To this end, observe that, for any $t \geq 0$,

$$M^P_{1,u}(tu) = \int_{\mathbb{R}^d} (\|z - tu\| - \|z\|)\, dP(z) - t.$$

Straightforward computations entail that, for all $z \in \mathbb{R}^d$,

$$\|z - tu\| - \|z\| - t = -\frac{2t(u'z + \|z\|)}{\|z - tu\| + \|z\| + t} \to -(u'z + \|z\|)$$

as $t \to \infty$. Because $\|z - tu\| - \|z\| - t \leq 0$ for all $z \in \mathbb{R}^d$ and $t \geq 0$, Fatou's lemma entails that

$$\limsup_{t\to\infty} M^P_{1,u}(tu) \leq -\int_{\mathbb{R}^d} (u'z + \|z\|)\, dP(z) = m^*. \tag{C.32}$$

If $m^* = -\infty$, then the infimum value of $M^P_{1,u}$ over $\mathbb{R}^d$ is $-\infty (= m^*)$. If $m^* > -\infty$, then observe that, for all $\mu \in \mathbb{R}^d$,

$$M^P_{1,u}(\mu) - m^* = \int_{\mathbb{R}^d} (\|z - \mu\| + u'(z - \mu))\, dP(z) \geq 0. \tag{C.33}$$

It follows that, in all cases, m^* is indeed the infimum value of $M^P_{1,u}$ over $\mathbb{R}^d$.

Let us now prove the equivalence in the statement. Assume that $M^P_{1,u}$ admits a minimum over $\mathbb{R}^d$. This implies that m^* is finite. Fix $\mu_0 \in \mathbb{R}^d$ such that $M^P_{1,u}(\mu_0) = m^*$. Then, (C.33) entails that $\|z - \mu_0\| + u'(z - \mu_0) = 0$ P-almost surely. This yields that P is supported on a halfline with direction $-u$.

Assume now that P is supported on the halfline $\mathcal{L} = \{z_0 - \lambda u : \lambda \geq 0\}$. For the moment, further assume that the integral $\int_{\mathbb{R}^d} (\|z\| + u'z)\, dP(z)$ is finite. In particular, m^* is finite as well. Thus, (C.33) entails that $M^P_{1,u}$ attains its infimum value m^* at any point μ such that $\|z - \mu\| + u'(z - \mu) = 0$ P-almost surely. In particular, this happens for any $\mu \in \{z_0 + \lambda u : \lambda \geq 0\}$. It follows that $M^P_{1,u}$ admits a minimum. Therefore, it remains to show that $\int_{\mathbb{R}^d} (\|z\| + u'z)\, dP(z)$ is finite. For this purpose, it is enough to show

$$\sup_{z\in\mathcal{L}} (\|z\| + u'z) < \infty. \tag{C.34}$$

For any $z \in \mathcal{L}$, say, $z := z_0 - \lambda u$, write

$$\|z\| + u'z = \|z_0 - \lambda u\| - \lambda + u'z_0 =: \phi(\lambda).$$

For any $\lambda > 0$, we have

$$\|z_0 - \lambda u\| - \lambda = \frac{\|z_0\|^2 - 2\lambda u'z_0}{\|z_0 - \lambda u\| + \lambda} \to -u'z_0,$$

as $\lambda \to \infty$. Consequently, we have $\lim_{\lambda\to+\infty} \phi(\lambda) = 0$. Because the map $\lambda \mapsto \phi(\lambda)$ is continuous over $[0, \infty)$, we deduce that ϕ is bounded over $[0, \infty)$. This establishes (C.34) and concludes the proof. □

We can now prove Theorem 4.1.

PROOF OF THEOREM 4.1. (i) Let us assume, *ad absurdum*, that the sequence $(\|\mu_{\alpha,u}(P_k)\|)$ does not diverge to ∞ as $k \to \infty$, so that it admits a bounded subsequence, (μ_n) say. We may thus fix $R > 0$ such that $\|\mu_n\| \leq R$ for any n. Letting (Q_n) denote the corresponding subsequence of (Q_k), observe that

$$\left|M^{Q_n}_{\tilde{\alpha},\tilde{u}}(\mu)\right| \leq (1 + \tilde{\alpha})\|\mu\|$$

for any $\mu \in \mathbb{R}^d$. On the one hand, the sequence $(M^{Q_n}_{\tilde{\alpha},u}(\mu_n))$ is bounded because (μ_n) is bounded. On the other hand, denoting by (P_n) the subsequence of (P_k) corresponding to (μ_n) and (Q_n), we have, for any $\mu \in \mathbb{R}^d$,

$$\begin{aligned}(1-c)M^{Q_n}_{\tilde{\alpha},\tilde{u}}(\mu_n) &\leq M^{P_n}_{\alpha,u}(\mu_n) + \Delta_n(R)\\ &\leq M^{P_n}_{\alpha,u}(\mu) + \Delta_n(R)\\ &\leq (1-c)M^{Q_n}_{\tilde{\alpha},\tilde{u}}(\mu) + \Delta_n(R) + \Delta_n(\|\mu\|),\end{aligned} \tag{C.35}$$

where we let, for any $r > 0$,

$$\Delta_n(r) := \sup_{z\in B_r} \big|M^{P_n}_{\alpha,u}(z) - (1-c)M^{Q_n}_{\tilde{\alpha},\tilde{u}}(z)\big|.$$

Since

$$M^{Q_n}_{\tilde{\alpha},\tilde{u}}(\mu) \leq \|\mu\|\Big(1 - \tilde{\alpha}\frac{\tilde{u}'\mu}{\|\mu\|}\Big)$$

for any $\mu \in \mathbb{R}^d$, we have

$$M^{Q_n}_{\tilde{\alpha},\tilde{u}}(s\tilde{u}) \leq s(1-\tilde{\alpha})$$

for any $s \geq 0$. Taking $\mu = R\tilde{u}$ in (C.35) thus yields

$$M^{Q_n}_{\tilde{\alpha},\tilde{u}}(\mu_n) \leq R(1-\tilde{\alpha}) + \frac{2\Delta_n(R)}{1-c}\cdot \tag{C.36}$$

Lemma C.2 provides

$$\Delta_n(R) \leq 2R\bigg\{\|w_n - w\| + \Lambda_n(B_R) + \frac{R}{T} + \Lambda_n(B_T)\bigg\} \tag{C.37}$$

for any $T > R$. Since $4/(1-c) \geq 1$, it follows from (C.36)–(C.37) that

$$M^{Q_n}_{\tilde{\alpha},\tilde{u}}(\mu_n) \leq \frac{4R}{1-c}\bigg\{(1-\tilde{\alpha}) + \|w_n - w\| + \Lambda_n(B_R) + \frac{R}{T} + \Lambda_n(B_T)\bigg\}.$$

Consequently, we have

$$\limsup_{n\to\infty} M^{Q_n}_{\tilde{\alpha},\tilde{u}}(\mu_n) \leq \frac{4R}{1-c}\bigg\{(1-\tilde{\alpha}) + \frac{R}{T}\bigg\}$$

for any $T > R$. Because R can be arbitrarily large and $\tilde{\alpha} > 1$, then letting $R, T \to \infty$ such that $R/T \to 0$, yields

$$\limsup_{n\to\infty} M^{Q_n}_{\tilde{\alpha},\tilde{u}}(\mu_n) = -\infty.$$

This contradicts the fact that $(M^{Q_n}_{\tilde{\alpha},\tilde{u}}(\mu_n))$ is bounded.

(ii) Consider first the case under which Q is not supported on a halfline with direction $-\tilde{u}$. Assume, *ad absurdum*, that the sequence $(\|\mu_{\alpha,u}(P_k)\|)$ does not diverge to ∞ as $k \to \infty$. It thus admits a bounded subsequence, (μ_n) say. Up to extracting a further subsequence, we may assume that (μ_n) converges to some $\mu_0 \in \mathbb{R}^d$. Denoting by (P_n) the corresponding subsequence of (P_k), Lemmas C.2–C.3 entail that, for any $\mu \in \mathbb{R}^d$,

$$(1-c)M^Q_{1,\tilde{u}}(\mu_n) = M^{P_n}_{\alpha,u}(\mu_n) + o(1) \leq M^{P_n}_{\alpha,u}(\mu) + o(1) = (1-c)M^Q_{1,\tilde{u}}(\mu) + o(1).$$

Taking limits as $n \to \infty$ in both sides, continuity of $M^Q_{1,\tilde{u}}$ over $\mathbb{R}^d$ yields

$$M^Q_{1,\tilde{u}}(\mu_0) \leq M^Q_{1,\tilde{u}}(\mu)$$

for any $\mu \in \mathbb{R}^d$. It follows that μ_0 is a minimizer of $M^Q_{1,\tilde{u}}$. Because Q is not supported on a halfline with direction $-\tilde{u}$, Lemma C.4 entails that $M_{1,\tilde{u}}(Q)$ admits no minimum over $\mathbb{R}^d$, a contradiction. We deduce that $\|\mu_{\alpha,u}(P_k)\| \to \infty$ as $k \to \infty$.

Second, consider the case under which there exists a bounded sequence (z_k) such that, for all k, Q_k is supported on the halfline $\{z_k - \lambda\tilde{u} : \lambda \geq 0\}$ and Λ_k is supported on the halfline $\{z_k + \lambda\tilde{u} : \lambda \geq r_k\}$ with $r_k \to \infty$. Because $\|z_k\|/r_k \to 0$, the support of Λ_k does not contain the origin of $\mathbb{R}^d$ for sufficiently large values of k. Denoting then by $\Lambda_{z_k,\tilde{u}}$ the distribution of $\tilde{u}'(Y - z_k)$ when Y has distribution Λ_k, we have

$$\int_{\mathbb{R}^d\setminus\{0\}} \frac{z}{\|z\|}\, d\Lambda_k(z) = \int_{r_k}^{\infty} \frac{z_k + \lambda\tilde{u}}{\|z_k + \lambda\tilde{u}\|}\, d\Lambda_{z_k,\tilde{u}}(\lambda).$$

Thus, Lemma A.1 entails that

$$\begin{aligned}
&\left\| \int_{\mathbb{R}^d\setminus\{0\}} \frac{z}{\|z\|}\, d\Lambda_k(z) - \tilde{u} \right\| \\
&\quad = \left\| \int_{r_k}^{\infty} \left\{ \frac{z_k + \lambda\tilde{u}}{\|z_k + \lambda\tilde{u}\|} - \frac{\lambda\tilde{u}}{\|\lambda\tilde{u}\|} \right\} d\Lambda_{z_k,\tilde{u}}(\lambda) \right\| \\
&\quad \leq 2\|z_k\| \int_{r_k}^{\infty} \frac{1}{\lambda}\, d\Lambda_{z_k,\tilde{u}}(\lambda) \\
&\quad \leq \frac{2\|z_k\|}{r_k}\cdot
\end{aligned}$$

Because $\|z_k\|/r_k \to 0$, this entails that $w = \tilde{u}$. Recalling that

$$\tilde{u} = \frac{\alpha u + cw}{\|\alpha u + cw\|},$$

this yields that $u = \tilde{u}$. Since $\tilde{\alpha} = 1$, we thus have $c = (1-\alpha)/2$. Because P_k is supported on a line with direction u, Theorem 1 from [27] entails that any spatial quantile of order α in direction u for P_k writes as $z_k + \ell_\alpha u$, where ℓ_α is a (univariate) spatial quantile of order α in direction 1 for $P_{z_k,u}$ (equivalently, a standard quantile of order $(1+\alpha)/2$ for $P_{z_k,u}$), with $P_{z_k,u}$ the distribution of $u'(Z - z_k)$ when Z has distribution P_k. This implies that the set $S_{\alpha,u}(P_k)$ of spatial quantiles of order α in direction u for P_k satisfies

$$S_{\alpha,u}(P_k) \subset \{z_k + \lambda u : \lambda \in \mathbb{R}\}.$$

Because $P_{z_k,u}[(-\infty,\lambda]] = 1 - c = (1+\alpha)/2$ for any $\lambda \in [0, r_k)$, we have

$$\{z_k + \lambda u : 0 \leq \lambda < r_k\} \subset S_{\alpha,u}(P_k).$$

Since (Q_k) is tight, there exists $R > 0$ such that

$$\inf_k Q_k(B_R) > 0.$$

In particular, since (z_k) is bounded, there exists $r^* > 0$ such that

$$\inf_k Q_k\Big(\{z_k + \lambda u : -r^* < \lambda \leq 0\}\Big) > 0.$$

This implies that $P_{z_k,u}[(-\infty, -r^*]] < 1 - c = (1+\alpha)/2$ for any k, hence that

$$\{z_k + \lambda u : 0 \leq \lambda < r_k\} \subset S_{\alpha,u}(P_k) \subset \{z_k + \lambda u : -r^* < \lambda < \infty\}.$$

Therefore, $\mu_{\alpha,u}(P_k)$, which, by definition, is the barycentre of $S_{\alpha,u}(P_k)$, satisfies

$$\|\mu_{\alpha,u}(P_k) - (z_k - r^* u)\| \geq \frac{r_k + r^*}{2}\cdot$$

It follows that $\|\mu_{\alpha,u}(P_k)\| \to \infty$ as $k \to \infty$.

(iii) Let us first show that $(\mu_{\alpha,u}(P_k))$ is bounded. To do that, we start by showing that there exist $R>0$ and $\kappa>0$ such that

$$M^{Q_k}_{\tilde\alpha,\tilde u}(\mu)>\kappa\|\mu\|$$

for all k and $\mu\in\mathbb{R}^d\setminus B_R$. We have

$$\frac{M^{Q_k}_{\tilde\alpha,\tilde u}(\mu)}{\|\mu\|}=\int_{\mathbb{R}^d}\left(\frac{\|z-\mu\|-\|z\|}{\|\mu\|}\right)dQ_k(z)-\tilde\alpha\frac{\tilde u'\mu}{\|\mu\|}.$$

Fix $\eta\in(0,1/2)$ small enough to have $(1+\tilde\alpha)(1+2\eta)/2<1$. Because (Q_k) converges weakly, it is tight. In particular, there exists $R_0>0$ such that

$$\inf_k Q_k(B_{R_0})>\frac{1+\tilde\alpha}{2}(1+2\eta).$$

Write then

$$\begin{aligned}
&\int_{\mathbb{R}^d}\left(\frac{\|z-\mu\|-\|z\|}{\|\mu\|}\right)dQ_k(z)\\
&\quad=\int_{B_{\eta\|\mu\|}}\left(\frac{\|\mu\|-2\mu'z/\|\mu\|}{\|z-\mu\|+\|z\|}\right)dQ_k(z)\\
&\qquad\qquad+\int_{\mathbb{R}^d\setminus B_{\eta\|\mu\|}}\left(\frac{\|z-\mu\|-\|z\|}{\|\mu\|}\right)dQ_k(z)\\
&\quad\geq\int_{B_{\eta\|\mu\|}}\left(\frac{\|\mu\|-2\|z\|}{\|\mu\|+2\|z\|}\right)dQ_k(z)-Q_k\big(\mathbb{R}^d\setminus B_{\eta\|\mu\|}\big)\\
&\quad\geq\frac{1-2\eta}{1+2\eta}\,Q_k(B_{\eta\|\mu\|})-Q_k\big(\mathbb{R}^d\setminus B_{\eta\|\mu\|}\big)\\
&\quad=\frac{2}{1+2\eta}\,Q_k(B_{\eta\|\mu\|})-1.
\end{aligned}$$

Letting $R:=R_0/\eta$, it follows that, for any $\mu\in\mathbb{R}^d\setminus B_R$, we have

$$\frac{M^{Q_k}_{\tilde\alpha,\tilde u}(\mu)}{\|\mu\|}\geq\frac{2}{1+2\eta}\Big(Q_k(B_{R_0})-\frac{1+\tilde\alpha}{2}(1+2\eta)\Big)$$

for all k. Letting

$$\kappa:=\frac{2}{1+2\eta}\Big(\inf_k Q_k(B_{R_0})-\frac{1+\tilde\alpha}{2}(1+2\eta)\Big)>0,$$

we then have

$$M^{Q_k}_{\tilde\alpha,\tilde u}(\mu)\geq\kappa\|\mu\|$$

for all k and $\mu\in\mathbb{R}^d\setminus B_R$. In particular, for any $\mu\in\mathbb{R}^d$ with $\|\mu\|=R$, we have

$$\begin{aligned}
M^{P_k}_{\alpha,u}(\mu)&\geq(1-c)M^{Q_k}_{\tilde\alpha,\tilde u}(\mu)-\sup_{\mu\in B_{2R}}\big|M^{P_k}_{\alpha,u}(\mu)-(1-c)M^{Q_k}_{\tilde\alpha,\tilde u}(\mu)\big|\\
&\geq(1-c)\kappa R-\sup_{\mu\in B_{2R}}\big|M^{P_k}_{\alpha,u}(\mu)-(1-c)M^{Q_k}_{\tilde\alpha,\tilde u}(\mu)\big|\\
&>\frac{(1-c)\kappa R}{2}
\end{aligned}$$

for all k large enough. In particular, we have $M^{P_k}_{\alpha,u}(\mu) > 0$ for all $\mu \in \mathbb{R}^d$ with $\|\mu\| = R$ and all k large enough. Because $M^{P_k}_{\alpha,u}(0) = 0$, the convexity of $M^{P_k}_{\alpha,u}$ entails that, for all k large enough, we have $M^{P_k}_{\alpha,u}(\mu) > 0$ for all $\mu \in \mathbb{R}^d$ such that $\|\mu\| \geq R$. Since $\mu_{\alpha,u}(P_k)$ is a minimizer of $M^{P_k}_{\alpha,u}$, we must have $\mu_{\alpha,u}(P_k) \in B_R$ for all k large enough. This shows that the sequence $(\mu_{\alpha,u}(P_k))$ is bounded.

Now assume that Q is not supported on a line with direction $\tilde{u}$ and let us show that $\mu_{\alpha,u}(P_k) \to \mu_{\tilde{\alpha},\tilde{u}}(Q)$ as $k \to \infty$. To this end, consider an arbitrary subsequence of $(\mu_{\alpha,u}(P_k))$. Because it is bounded, it admits a further subsequence (μ_n) converging in $\mathbb{R}^d$, say to μ_0. Denoting by (P_n) the corresponding subsequence of (P_k), Lemmas C.2–C.3 entail that, for any $\mu \in \mathbb{R}^d$, we have

$$(1-c)M^Q_{\tilde{\alpha},\tilde{u}}(\mu_n) = M^{P_n}_{\alpha,u}(\mu_n) + o(1) \leq M^{P_n}_{\alpha,u}(\mu) + o(1) = (1-c)M^Q_{\tilde{\alpha},\tilde{u}}(\mu) + o(1).$$

Taking limits as $n \to \infty$ in both sides, the continuity of $M^Q_{\tilde{\alpha},\tilde{u}}$ over $\mathbb{R}^d$ yields

$$M^Q_{\tilde{\alpha},\tilde{u}}(\mu_0) \leq M^Q_{\tilde{\alpha},\tilde{u}}(\mu)$$

for any $\mu \in \mathbb{R}^d$. It follows that μ_0 is a minimizer of $M^Q_{\tilde{\alpha},\tilde{u}}$. Because Q is not concentrated on a line with direction $\tilde{u}$, Theorem 1 in [27] entails that Q admits a unique minimizer of order $\tilde{\alpha}$ in direction $\tilde{u}$, namely $\mu_{\tilde{\alpha},\tilde{u}}(Q)$. This implies that $\mu_0 = \mu_{\tilde{\alpha},\tilde{u}}(Q)$. We thus proved that any subsequence of $(\mu_{\alpha,u}(P_k))$ admits a further subsequence converging to $\mu_{\tilde{\alpha},\tilde{u}}(Q)$. We conclude that $\mu_{\alpha,u}(P_k) \to \mu_{\tilde{\alpha},\tilde{u}}(Q)$ as $k \to \infty$. □

Finally, we provide an alternative proof of Corollary 2.2 based on Theorem 4.1.

PROOF OF COROLLARY 2.2. First, let us show that

$$\text{BDP}(\mu_{\alpha,u}; x_1, \ldots, x_n) \leq \left\lceil \frac{n(1-\alpha)}{2} \right\rceil / n. \tag{C.38}$$

Fix $\ell = \lceil n(1-\alpha)/2 \rceil$. Letting $m := \frac{1}{n-\ell}\sum_{i=\ell+1}^n x_i$ and denoting as δ_x the Dirac probability measure at x, consider the sequence of probability measures

$$P_k := \frac{1}{n}\sum_{i=1}^n \delta_{y_i^{(k)}},$$

where $y_i^{(k)} = m + ku$ for $i = 1, \ldots, \ell$ and $y_i^{(k)} = x_i$ for $i = \ell+1, \ldots, n$. Define

$$Q_k := \frac{1}{n-\ell}\sum_{i=\ell+1}^n \delta_{y_i^{(k)}} \quad \text{and} \quad \Lambda_k := \frac{1}{\ell}\sum_{i=1}^{\ell} \delta_{y_i^{(k)}}$$

for all k. Letting $c := \ell/n$, we then have $P_k = (1-c)Q_k + c\Lambda_k$. Obviously, $\Lambda_k(B_r) \to 0$ for any $r > 0$ as $k \to \infty$. Using the notation of Theorem 4.1, we have $w = u$, hence

$$\tilde{\alpha} = \frac{\alpha + \ell/n}{1 - \ell/n} = \frac{n\alpha + \lceil n(1-\alpha)/2 \rceil}{n - \lceil n(1-\alpha)/2 \rceil}.$$

We consider two cases. (A) $n(1-\alpha)/2$ is not an integer. Then, $\tilde{\alpha} > 1$, so that Theorem 4.1(i) implies that $\|\mu_{\alpha,u}(P_k)\| \to \infty$, which establishes (C.38). (B) $n(1-\alpha)/2$ is an integer. Then, $\tilde{\alpha} = 1$. Of course, the sequence (Q_k) converges weakly to $Q := Q_1$ since $Q_k = Q_1$ for any k. If $x_{\ell+1}, \ldots, x_n$ do not all belong to a halfline with direction $-\tilde{u} = -u$, then Part a) of Theorem 4.1(ii) implies that $\|\mu_{\alpha,u}(P_k)\| \to \infty$, which establishes (C.38). If $x_{\ell+1}, \ldots, x_n$ all belong to a halfline with direction $-u$, then applying Part b) of Theorem 4.1(ii) with $z_k \equiv m$ implies that $\|\mu_{\alpha,u}(P_k)\| \to \infty$, which establishes again (C.38).

Now, let us show that

$$\text{BDP}(\mu_{\alpha,u}; x_1, \ldots, x_n) \geq \left\lceil \frac{n(1-\alpha)}{2} \right\rceil / n. \tag{C.39}$$

To this end, fix an arbitrary $\ell \in \{1, \ldots, n\}$ and assume that there exists a sequence $(\{y_1^{(k)}, \ldots, y_n^{(k)}\})_k \subset \mathbb{R}^d$ of samples such that $y_i^{(k)} \in \{x_1, \ldots, x_n\}$ for all $i = \ell+1, \ldots, n$ and all k, and

$$\|\mu_{\alpha,u}(x_1, \ldots, x_n) - \mu_{\alpha,u}(y_1^{(k)}, \ldots, y_n^{(k)})\| \to \infty$$

as $k \to \infty$. In particular, we have

$$\|\mu_{\alpha,u}(y_1^{(k)}, \dots, y_n^{(k)})\| \to \infty$$

as $k \to \infty$. For all k, define the probability measure

$$P_k = \frac{1}{n} \sum_{i=1}^n \delta_{y_i^{(k)}}.$$

Up to extracting subsequences, let us assume that each sequence $(y_i^{(k)})_k$ either converges in $\mathbb{R}^d$ or exits any compact as $k \to \infty$. Then, define

$$B := \left\{ i \in \{1, \dots, n\} : (y_i^{(k)})_k \text{ converges in } \mathbb{R}^d \right\}$$

and $U := \{1, \dots, n\} \setminus B$. For any k, let

$$Q_k = \frac{1}{|B|} \sum_{i \in B} \delta_{y_i^{(k)}}, \quad \Lambda_k = \frac{1}{|U|} \sum_{i \in U} \delta_{y_i^{(k)}} \quad \text{and} \quad w_k := \int_{\mathbb{R}^d} \frac{z}{\|z\|} \, d\Lambda_k(z).$$

Since (w_k) is contained in the closed unit ball of $\mathbb{R}^d$, up to extracting a further subsequence we may assume that (w_k) converges to some $w \in \mathbb{R}^d$ with $\|w\| \leq 1$. Letting $c := |U|/n$, we have $P_k = (1-c)Q_k + c\Lambda_k$, where (Q_k) converges weakly and $\Lambda_k(B_r) \to 0$ for any $r > 0$ as $k \to \infty$. Therefore, Theorem 4.1(ii) entails that $\|\alpha u + cw\| \geq 1 - c$. In particular, because $\|w\| \leq 1$, we have $\alpha + c \geq 1 - c$, hence $c \geq (1-\alpha)/2$. Now observe that, because $(y_i^{(k)})$ is bounded for all $i = \ell + 1, \dots, n$, we have $|U| \leq \ell$, i.e. $c \leq \ell/n$. This implies that $\ell \geq n(1-\alpha)/2$. Because $\ell \in \mathbb{N}$, this yields $\ell \geq \lceil n(1-\alpha)/2 \rceil$. This establishes (C.39). □

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