Lecture Notes for STAT-F404

6 Hypothesis Testing: UMPU Tests

6.1 Unbiasedness and similarity

Let $\mathcal{H}_0$ and $\mathcal{H}_1$ characterize a testing problem in a parametric family indexed by $\boldsymbol{\theta} \in \Theta \subset \mathbb{R}^k$. Recall that a test ϕ is an α-level test if $\mathrm{E}_{\boldsymbol{\theta}}[\phi] \leq \alpha$ for any $\boldsymbol{\theta} \in \mathcal{H}_0$. It is not unnatural to require that a test rejects more often under $\mathcal{H}_1$ than under $\mathcal{H}_0$, which leads to the concept of *unbiasedness*.

Definition 1. *A test ϕ for $\mathcal{H}_0$ against $\mathcal{H}_1$ is* unbiased at level α *if (i) ϕ is an α-level test and (ii) $\mathrm{E}_{\boldsymbol{\theta}}[\phi] \geq \alpha$ for any $\boldsymbol{\theta} \in \mathcal{H}_1$.*

The unbiasedness principle then consists in restricting to tests that are unbiased at level α: a test ϕ^* will be said to be *uniformly most powerful unbiased* (UMPU) at level α if (a) ϕ^* is unbiased at level α, and (b) for any ϕ that is unbiased at level α, one has $\mathrm{E}_{\boldsymbol{\theta}}[\phi^*] \geq \mathrm{E}_{\boldsymbol{\theta}}[\phi]$ for any $\boldsymbol{\theta} \in \mathcal{H}_1$.

Now, if the size/power function $\boldsymbol{\theta} \mapsto \mathrm{E}_{\boldsymbol{\theta}}[\phi]$ is continuous (as it is for any test in the framework of exponential families), then a necessary condition for unbiasedness is *similarity*.

Definition 2. *A test ϕ for $\mathcal{H}_0$ against $\mathcal{H}_1$ is* similar at level α *(or* α-similar*) if $\mathrm{E}_{\boldsymbol{\theta}}[\phi] = \alpha$ for any $\boldsymbol{\theta} \in \bar{\mathcal{H}}$, where $\bar{\mathcal{H}} := \mathrm{adh}(\mathcal{H}_0) \cap \mathrm{adh}(\mathcal{H}_1)$ is the boundary between $\mathcal{H}_0$ and $\mathcal{H}_1$.*

Parallel as above, we say that a test ϕ^* is uniformly most powerful in the class of α-similar tests if (a) ϕ^* is similar at level α, and (b) for any ϕ that is α-similar, one has $\mathrm{E}_{\boldsymbol{\theta}}[\phi^*] \geq \mathrm{E}_{\boldsymbol{\theta}}[\phi]$ for any $\boldsymbol{\theta} \in \mathcal{H}_1$. As an exercise, the reader can prove that the following statement holds as soon as $\boldsymbol{\theta} \mapsto \mathrm{E}_{\boldsymbol{\theta}}[\phi]$ is continuous: if ϕ^* is uniformly most powerful in the class of α-similar tests and if ϕ^* is an α-level test, then ϕ^* is UMPU at level α.

6.2 Two-sided testing problems in exponential families

Let $(\mathcal{X}, \mathcal{A}, \mathcal{P} = \{\mathrm{P}_\theta : \theta \in \Theta \subset \mathbb{R}\})$ be an exponential model indexed by a scalar parameter θ (here, Θ is an interval of $\mathbb{R}$, or $\mathbb{R}$ itself). Recall that, after appropriately choosing the σ-finite dominating measure μ at hand, this implies that the corresponding densities take the form

$$f_\theta(x) = C(\theta)\exp(\theta T(x)),$$

and that, when $\mathbf{X} \sim \mathrm{P}_\theta$, the distribution of T admits the density

$$f_\theta^T(t) = C(\theta)\exp(\theta t)$$

with respect to the induced dominating measure μ^T. In this framework, consider the two-sided problem

$$H_0 : \theta = \theta_0 \quad \text{vs} \quad H_1 : \theta \neq \theta_0,$$

where θ_0 is a fixed value in the interior of Θ. Since a UMP test at level α cannot exist for this problem (why?), we are after a UMPU test at level α.

Theorem 1. *In the exponential model above, fix $\alpha \in (0,1)$ and $\theta_0 \in \mathrm{int}(\Theta)$. Then, (i) there exist $\gamma_{1,\alpha}, \gamma_{2,\alpha} \in [0,1]$ and $t_{1,\alpha}, t_{2,\alpha} \in \mathbb{R}$ with $t_{1,\alpha} \leq t_{2,\alpha}$ such that the test defined by*

$$\phi_\alpha^*(\mathbf{x}) = \begin{cases} 1 & \text{if } T(\mathbf{x}) \notin [t_{1,\alpha}, t_{2,\alpha}] \\ \gamma_{1,\alpha} & \text{if } T(\mathbf{x}) = t_{1,\alpha} \\ \gamma_{2,\alpha} & \text{if } T(\mathbf{x}) = t_{2,\alpha} \\ 0 & \text{if } T(\mathbf{x}) \in (t_{1,\alpha}, t_{2,\alpha}) \end{cases}$$

satisfies $\mathrm{E}_{\theta_0}[\phi_\alpha^*] = \alpha$ and $\mathrm{E}_{\theta_0}[\phi_\alpha^* T] = \alpha \mathrm{E}_{\theta_0}[T]$. (ii) The test ϕ_α^* is UMPU at level α for the problem of testing $H_0 : \theta = \theta_0$ against $H_1 : \theta \neq \theta_0$.

We now give an interpretation for the constraint $\mathrm{E}_{\theta_0}[\phi^*_\alpha T] = \alpha \mathrm{E}_{\theta_0}[T]$. Note that we have

$$\begin{aligned}\frac{d}{d\theta}\mathrm{E}_\theta[\phi] &= \int_{\mathcal{X}} \phi(x)\frac{d}{d\theta}\Big(C(\theta)\exp(\theta T(x))\Big)\,d\mu(x)\\ &= \frac{C'(\theta)}{C(\theta)}\mathrm{E}_\theta[\phi] + \mathrm{E}_\theta[\phi T],\end{aligned} \tag{6.1}$$

which, for $\phi \equiv 1$, yields $\mathrm{E}_\theta[T] = -C'(\theta)/C(\theta)$. Using this in (6.1) provides

$$\frac{d}{d\theta}\mathrm{E}_\theta[\phi] = \mathrm{E}_\theta[\phi T] - \mathrm{E}_\theta[\phi]\mathrm{E}_\theta[T] \quad (= \mathrm{Cov}_\theta[\phi, T]).$$

The constraint $\mathrm{E}_{\theta_0}[\phi^*_\alpha T] = \alpha \mathrm{E}_{\theta_0}[T]$ in the theorem above may then be interpreted as

$$\frac{d}{d\theta}\mathrm{E}_\theta[\phi^*_\alpha]|_{\theta=\theta_0} = 0.$$

In the exponential family considered (where the size/power function of any test is smooth), the constraints $\mathrm{E}_{\theta_0}[\phi^*_\alpha] = \alpha$ and $\mathrm{E}_{\theta_0}[\phi^*_\alpha T] = \alpha \mathrm{E}_{\theta_0}[T]$ thus clearly are necessary conditions for the test ϕ^*_α to be unbiased at level α. This will play a role in the proof of the theorem.

Proof. (i) Consider the induced model $(\mathcal{T}, \mathcal{B}, \mathcal{P}^T = \{\mathrm{P}^T_\theta : \theta \in \Theta\})$, which, as recalled above, is dominated by the induced measure μ^T, leading to the corresponding densities $f^T_\theta(t) = C(\theta)\exp(\theta t)$. In this induced model, fix $\theta_1 > \theta_0$ arbitrarily, and consider the problem of testing $\mathcal{H}_0 : \theta = \theta_0$ against $\mathcal{H}_1 : \theta = \theta_1$ in the class of tests $\varphi = \varphi(T)$ satisfying

$$\begin{aligned}\int_{\mathcal{T}} \varphi(t) f^T_{\theta_0}(t)\,d\mu^T(x) &= \mathrm{E}_{\theta_0}[\varphi]\\ &= \alpha\end{aligned} \tag{6.2}$$

and

$$\begin{aligned}\int_{\mathcal{T}} \varphi(t) t f^T_{\theta_0}(t)\,d\mu^T(x) &= \mathrm{E}_{\theta_0}[\varphi T]\\ &= \alpha \mathrm{E}_{\theta_0}[T].\end{aligned} \tag{6.3}$$

For this problem, consider

$$\mathcal{M} = \{(\mathrm{E}_{\theta_0}[\varphi], \mathrm{E}_{\theta_0}[\varphi T]) : \varphi \text{ a test}\}.$$

For any u in a neighborhood $[\alpha - \varepsilon, \alpha + \varepsilon]$ of α, the UMP test, φ_u^+ say, for $\mathcal{H}_0 : \theta \leq \theta_0$ against $\mathcal{H}_1 : \theta > \theta_0$ at level u provides $(\mathrm{E}_{\theta_0}[\varphi_u^+], \mathrm{E}_{\theta_0}[\varphi_u^+ T]) = (u, c_u^+)$, with $c_u^+ > u\mathrm{E}_{\theta_0}[T]$ (since $\frac{d}{d\theta}\mathrm{E}_\theta[\varphi_u^+]|_{\theta=\theta_0} = \mathrm{E}_{\theta_0}[\varphi_u^+ T] - u\mathrm{E}_{\theta_0}[T] > 0$; see the remark at the end of Section 5.3). Similarly, for any $u \in [\alpha - \varepsilon, \alpha + \varepsilon]$, the UMP test, ϕ_u^- say, for $\mathcal{H}_0 : \theta \geq \theta_0$ against $\mathcal{H}_1 : \theta < \theta_0$ at level u provides $(\mathrm{E}_{\theta_0}[\varphi_u^-], \mathrm{E}_{\theta_0}[\varphi_u^- T]) = (u, c_u^-)$, with $c_u^- < u\mathrm{E}_{\theta_0}[T]$. Jointly with the convexity of $\mathcal{M}$, this implies that $(\alpha, \alpha\mathrm{E}_{\theta_0}[T])$ is an interior point of $\mathcal{M}$. Part (iv) of the second version of the generalized Neyman-Pearson lemma thus implies that there exist $k_1, k_2 \in \mathbb{R}$ and a measurable function $\gamma(\cdot)$ such that the test φ defined by

$$\varphi(t) = \begin{cases} 1 & \text{if } f_{\theta_1}^T(t) > k_1 f_{\theta_0}^T(t) + k_2 t f_{\theta_0}^T(t) \\ \gamma(t) & \text{if } f_{\theta_1}^T(t) = k_1 f_{\theta_0}^T(t) + k_2 t f_{\theta_0}^T(t) \\ 0 & \text{if } f_{\theta_1}^T(t) < k_1 f_{\theta_0}^T(t) + k_2 t f_{\theta_0}^T(t) \end{cases}$$

satisfies (6.2)–(6.3). Using the explicit expression of $f_{\theta_1}^T$, this test rewrites

$$\varphi(t) = \begin{cases} 1 & \text{if } \exp((\theta_1 - \theta_0)t) > \ell_1 + \ell_2 t \\ \gamma(t) & \text{if } \exp((\theta_1 - \theta_0)t) = \ell_1 + \ell_2 t \\ 0 & \text{if } \exp((\theta_1 - \theta_0)t) < \ell_1 + \ell_2 t, \end{cases}$$

where we let $\ell_i := k_i C(\theta_0)/C(\theta_1)$, $i = 1, 2$. Since we cannot have $\varphi(t) = 1$ for any t (this would provide $\mathrm{E}_{\theta_0}[\varphi] = 1 \neq \alpha$, which would contradict (6.2)), nor either of

$$\varphi(t) = \begin{cases} 1 & \text{if } t > t_0 \\ \gamma & \text{if } t = t_0 \\ 0 & \text{if } t < t_0 \end{cases} \qquad \text{or} \qquad \varphi(t) = \begin{cases} 1 & \text{if } t < t_0 \\ \gamma & \text{if } t = t_0 \\ 0 & \text{if } t > t_0 \end{cases}$$

(this would provide $\frac{d}{d\theta}\mathrm{E}_\theta[\varphi]|_{\theta=\theta_0} = \mathrm{E}_{\theta_0}[\varphi T] - \alpha\mathrm{E}_{\theta_0}[T] > 0$ or < 0, respectively, which would contradict (6.3)), we must then have

$$\varphi(t) = \begin{cases} 1 & \text{if } t \notin [t_1, t_2] \\ \gamma_1 & \text{if } t = t_1 \\ \gamma_2 & \text{if } t = t_2 \\ 0 & \text{if } t \in (t_1, t_2), \end{cases}$$

for some t_1, t_2 such that $t_1 \leq t_2$. It follows that the test ϕ defined by

$$\phi(\mathbf{x}) = \varphi(T(\mathbf{x})) = \begin{cases} 1 & \text{if } T(\mathbf{x}) \notin [t_1, t_2] \\ \gamma_1 & \text{if } T(\mathbf{x}) = t_1 \\ \gamma_2 & \text{if } T(\mathbf{x}) = t_2 \\ 0 & \text{if } T(\mathbf{x}) \in (t_1, t_2) \end{cases}$$

satisfies $\mathrm{E}_{\theta_0}[\phi] = \alpha$ and $\mathrm{E}_{\theta_0}[\phi T] = \alpha \mathrm{E}_{\theta_0}[T]$, which establishes the result.

(ii) Let ϕ^*_α be the test described in Part (i) of the theorem. Fix $\theta_1 \neq \theta_0$ arbitrarily. Then, it is possible (why?[1]) to find $\ell_1, \ell_2 \in \mathbb{R}$ such that

$$\phi^*_\alpha(\mathbf{x}) = \begin{cases} 1 & \text{if } \exp((\theta_1 - \theta_0)T(\mathbf{x})) > \ell_1 + \ell_2 T(\mathbf{x}) \\ \gamma_1 & \text{if } T(\mathbf{x}) = t_1 \\ \gamma_2 & \text{if } T(\mathbf{x}) = t_2 \\ 0 & \text{if } \exp((\theta_1 - \theta_0)T(\mathbf{x})) < \ell_1 + \ell_2 T(\mathbf{x}). \end{cases}$$

Thus,

$$\phi^*_\alpha(\mathbf{x}) = \begin{cases} 1 & \text{if } f_{\theta_1}(T(\mathbf{x})) > k_1 f_{\theta_0}(T(\mathbf{x})) + k_2 T(\mathbf{x}) f_{\theta_0}(T(\mathbf{x})) \\ \gamma_1 & \text{if } T(\mathbf{x}) = t_1 \\ \gamma_2 & \text{if } T(\mathbf{x}) = t_2 \\ 0 & \text{if } f_{\theta_1}(T(\mathbf{x})) < k_1 f_{\theta_0}(T(\mathbf{x})) + k_2 T(\mathbf{x}) f_{\theta_0}(T(\mathbf{x})) \end{cases}$$

for some $k_1, k_2 \in \mathbb{R}$. Part (ii) of the second version of the generalized Neyman-Pearson lemma thus entails that ϕ^*_α is most powerful for the problem of testing $H_0 : \theta = \theta_0$ against $H_1 : \theta = \theta_1$

[1] When answering this question, do not forget the case $t_{1,\alpha} = t_{2,\alpha}$

in the class of tests satisfying $\mathrm{E}_{\theta_0}[\phi] = \alpha$ and $\mathrm{E}_{\theta_0}[\phi T] = \alpha \mathrm{E}_{\theta_0}[T]$. Since ϕ^*_α does not depend on θ_1, it follows that ϕ^*_α is uniformly most powerful for the problem of testing $H_0 : \theta = \theta_0$ against $H_1 : \theta \neq \theta_0$ in the class of tests satisfying $\mathrm{E}_{\theta_0}[\phi] = \alpha$ and $\mathrm{E}_{\theta_0}[\phi T] = \alpha \mathrm{E}_{\theta_0}[T]$. Recalling that any unbiased test at level α must satisfy these constraints, ϕ^*_α is then UMPU at level α for the problem of testing $H_0 : \theta = \theta_0$ against $H_1 : \theta \neq \theta_0$ (unbiasedness of ϕ^*_α at level α follows by comparing its power function to that of the trivial test defined by $\phi(\mathbf{x}) = \alpha$ for any $\mathbf{x}$). □

If the distribution of T under P_{θ_0} is symmetric (automatically about $\mathrm{E}_{\theta_0}[T]$), then it is always possible to find $\gamma_\alpha \in [0, 1]$ and $h_\alpha \in \mathbb{R}^+$ such that the test ϕ^*_α defined by

$$\phi^*_\alpha(\mathbf{x}) = \begin{cases} 1 & \text{if } T(\mathbf{x}) \notin [\mathrm{E}_{\theta_0}[T] - h_\alpha, \mathrm{E}_{\theta_0}[T] + h_\alpha] \\ \gamma_\alpha & \text{if } T(\mathbf{x}) \in \{\mathrm{E}_{\theta_0}[T] - h_\alpha, \mathrm{E}_{\theta_0}[T] + h_\alpha\} \\ 0 & \text{if } T(\mathbf{x}) \in (\mathrm{E}_{\theta_0}[T] - h_\alpha, \mathrm{E}_{\theta_0}[T] + h_\alpha) \end{cases} \tag{6.4}$$

satisfies $\mathrm{E}_{\theta_0}[\phi^*_\alpha] = \alpha$ (this can be checked by rewriting this test as

$$\phi^*_\alpha(\mathbf{x}) = \begin{cases} 1 & \text{if } |T(\mathbf{x}) - \mathrm{E}_{\theta_0}[T]| > h_\alpha \\ \gamma_\alpha & \text{if } |T(\mathbf{x}) - \mathrm{E}_{\theta_0}[T]| = h_\alpha \\ 0 & \text{if } |T(\mathbf{x}) - \mathrm{E}_{\theta_0}[T]| < h_\alpha \end{cases}$$

and by proceeding as in the proof of Theorem 1(i) from the previous chapter). For this test, note that we have

$$\begin{aligned} \mathrm{E}_{\theta_0}[\phi^*_\alpha T] &= \mathrm{E}_{\theta_0}[\phi^*_\alpha (T - \mathrm{E}_{\theta_0}[T])] + \mathrm{E}_{\theta_0}[\phi^*_\alpha]\mathrm{E}_{\theta_0}[T] \\ &= 0 + \alpha \mathrm{E}_{\theta_0}[T] \\ &= \alpha \mathrm{E}_{\theta_0}[T], \end{aligned}$$

so that the second constraint is then automatically satisfied, which implies that this test is UMPU at level α. A typical example is obtained when testing $\mathcal{H}_0 : \mu = \mu_0$ against $\mathcal{H}_1 : \mu \neq \mu_0$ at level α on the basis of $\mathbf{X} = (X_1, \ldots, X_n)$, where the X_i's are i.i.d. Gaussian with unknown mean μ and fixed variance σ_0^2. If the distribution of T under P_{θ_0} is not symmetric, then both constraints need to be imposed (which, in practice, sometimes can only be achieved numerically); an example is obtained when testing $\mathcal{H}_0 : \sigma^2 = \sigma_0^2$ against $\mathcal{H}_1 : \sigma^2 \neq \sigma_0^2$ at level α on the basis of $\mathbf{X} = (X_1, \ldots, X_n)$, where the X_i's are i.i.d. Gaussian with fixed mean μ_0 and unknown variance σ^2.

6.3 Problems in exponential families with nuisance parameters

Let $(\mathcal{X}, \mathcal{A}, \mathcal{P} = \{\mathrm{P}_{\tau,\boldsymbol{\lambda}}\})$ be an exponential model with densities (with respect to a σ-finite dominating measure μ) of the form

$$f_{\tau,\boldsymbol{\lambda}}(x) = C(\tau, \boldsymbol{\lambda}) \exp(\tau T(x) + \boldsymbol{\lambda}'\mathbf{S}(x)),$$

with $\tau \in \mathbb{R}$ and $\boldsymbol{\lambda} \in \mathbb{R}^s$. In this framework, consider the one-sided problem

$$H_0 = \big\{\mathrm{P}_{\tau,\boldsymbol{\lambda}} : \tau \leq \tau_0\big\} \quad \text{vs} \quad H_1 = \big\{\mathrm{P}_{\tau,\boldsymbol{\lambda}} : \tau > \tau_0\big\} \tag{6.5}$$

at level α, where τ_0 is a fixed value and $\boldsymbol{\lambda}$ remains unspecified, hence plays the role of a nuisance parameter. Using the notation from Section 6.1, we have

$$\bar{\mathcal{H}} = \big\{\mathrm{P}_{\tau,\boldsymbol{\lambda}} : \tau = \tau_0\big\},$$

which is an exponential subfamily indexed by $\boldsymbol{\lambda}$ and with natural statistic $\mathbf{S}$. For the corresponding submodel, $\mathbf{S}$ is thus sufficient and complete.

Theorem 2. *Fix $\alpha \in (0,1)$. Then, we have the following in the framework above: (i) there exists a test of the form*

$$\phi_\alpha^*(\mathbf{x}) = \begin{cases} 1 & \textit{if } T(\mathbf{x}) > t(\mathbf{S}(x)) \\ \gamma(\mathbf{S}(x)) & \textit{if } T(\mathbf{x}) = t(\mathbf{S}(x)) \\ 0 & \textit{if } T(\mathbf{x}) < t(\mathbf{S}(x)) \end{cases}$$

satisfying

$$\mathrm{E}_{\tau_0,\boldsymbol{\lambda}}[\phi^*_\alpha|\mathbf{S}] = \alpha \quad \mathrm{P}_{\tau_0,\boldsymbol{\lambda}}\text{-}a.s.\ for\ any\ \boldsymbol{\lambda}. \tag{6.6}$$

(ii) This test is UMPU at level α for the testing problem (6.5).

A test that satisfies (6.6) is said to have *Neyman α-structure with respect to* $\mathbf{S}$ for the testing problem (6.5). Note that a test has Neyman α-structure with respect to $\mathbf{S}$ if and only if it is α-similar for the same problem (the necessary condition is trivial, whereas the sufficient one follows from sufficiency and completeness of $\mathbf{S}$ in the $\bar{\mathcal{H}}$-submodel). A corollary is that one may restrict to tests having Neyman α-structure with respect to $\mathbf{S}$ when looking for a UMPU test at level α, which makes condition (6.6) natural.

As an exercise, one can apply Theorem 2 to show that, when $X_1, \ldots, X_n$ are i.i.d. Gaussian with mean μ and variance σ^2, then

$$\phi^*_\alpha(\mathbf{x}) = \begin{cases} 1 & \text{if } ns^2/\sigma_0^2 > \chi^2_{n-1,1-\alpha} \\ 0 & \text{if } ns^2/\sigma_0^2 \leq \chi^2_{n-1,1-\alpha}, \end{cases}$$

where $s^2 = \frac{1}{n}\sum_{i=1}^n (X_i - \bar{X})^2$ and $\chi^2_{n-1,1-\alpha}$ is the $(1-\alpha)$-quantile of the χ^2_{n-1} distribution, is UMPU at level α for the problem of testing $\mathcal{H}_0 : \sigma^2 \leq \sigma_0^2$ against $\mathcal{H}_1 : \sigma^2 > \sigma_0^2$, and that, in the same model,

$$\phi^*_\alpha(\mathbf{x}) = \begin{cases} 1 & \text{if } \sqrt{n-1}(\bar{X} - \mu_0)/s > t_{n-1,1-\alpha} \\ 0 & \text{if } \sqrt{n-1}(\bar{X} - \mu_0)/s \leq t_{n-1,1-\alpha}, \end{cases}$$

where $t_{n-1,1-\alpha}$ is the $(1-\alpha)$-quantile of the t_{n-1} distribution, is UMPU at level α for the problem of testing $\mathcal{H}_0 : \mu \leq \mu_0$ against $\mathcal{H}_1 : \mu > \mu_0$ (the exercise is actually more complicated for this second testing problem[2]).

In the same general exponential framework as in the beginning of this section, one may also consider the two-sided problem

$$H_0 = \big\{\mathrm{P}_{\tau,\boldsymbol{\lambda}} : \tau = \tau_0\big\} \quad \text{vs} \quad H_1 = \big\{\mathrm{P}_{\tau,\boldsymbol{\lambda}} : \tau \neq \tau_0\big\} \tag{6.7}$$

[2]Do the change of variables $X_i \rightsquigarrow X_i - \mu_0$, which makes the null hypothesis become $\mathcal{H}_0 : \mu \leq 0$, and use the fact that $\bar{X}/s = \bar{X}/(\frac{1}{n}\sum_i X_i^2 - \bar{X}^2)^{1/2}$ is an increasing function of $\bar{X}$ for fixed $\sum_i X_i^2$.

at level α, where τ_0 is still a fixed value and $\boldsymbol{\lambda}$ remains unspecified—which yields again $\bar{\mathcal{H}} = \{\mathrm{P}_{\tau,\boldsymbol{\lambda}} : \tau = \tau_0\}$. We have the following analog of Theorem 2.

Theorem 3. *Fix $\alpha \in (0,1)$. Then, we have the following in the framework above: (i) there exists a test of the form*

$$\phi^*_\alpha(\mathbf{x}) = \begin{cases} 1 & \textit{if } T(\mathbf{x}) \notin [t_1(\mathbf{S}(x)), t_2(\mathbf{S}(x))] \\ \gamma_1(\mathbf{S}(x)) & \textit{if } T(\mathbf{x}) = t_1(\mathbf{S}(x)) \\ \gamma_2(\mathbf{S}(x)) & \textit{if } T(\mathbf{x}) = t_2(\mathbf{S}(x)) \\ 0 & \textit{if } T(\mathbf{x}) \in (t_1(\mathbf{S}(x)), t_2(\mathbf{S}(x))) \end{cases}$$

satisfying both

$$\mathrm{E}_{\tau_0,\not{\boldsymbol{\lambda}}}[\phi^*_\alpha|\mathbf{S}] = \alpha \quad \mathrm{P}_{\tau_0,\boldsymbol{\lambda}}\textit{-a.s. for any } \boldsymbol{\lambda}$$

and

$$\mathrm{E}_{\tau_0,\not{\boldsymbol{\lambda}}}[\phi^*_\alpha T|\mathbf{S}] = \alpha \mathrm{E}_{\tau_0,\not{\boldsymbol{\lambda}}}[\phi^*_\alpha|\mathbf{S}] \quad \mathrm{P}_{\tau_0,\boldsymbol{\lambda}}\textit{-a.s. for any } \boldsymbol{\lambda}.$$

(ii) This test is UMPU at level α for the testing problem (6.7).

Coming back to the Gaussian model considered above, one may then show in particular that the test defined by

$$\phi^*_\alpha(\mathbf{x}) = \begin{cases} 1 & \text{if } |\sqrt{n-1}(\bar{X} - \mu_0)/s| > t_{n-1,1-(\alpha/2)} \\ 0 & \text{if } |\sqrt{n-1}(\bar{X} - \mu_0)/s| \leq t_{n-1,1-(\alpha/2)} \end{cases}$$

is UMPU at level α for the problem of testing $\mathcal{H}_0 : \mu = \mu_0$ against $\mathcal{H}_1 : \mu \neq \mu_0$.